3.4
Product Construction

Union and Intersection

Question: Are languages accepted by DFAs closed under union? That is, given DFAs M_1 and M_2 is there a DFA that accepts $L(M_1) \cup L(M_2)$?
How about intersection $L(M_1) \cap L(M_2)$?

Idea from programming: on input string w

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- Solution: Simulate M_1 and M_2 in parallel by keeping track of states of both machines

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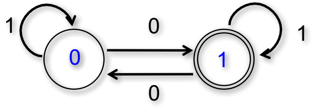
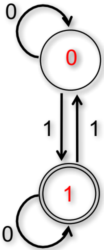


M_1 accepts #0 = odd



M_2 accepts #1 = odd

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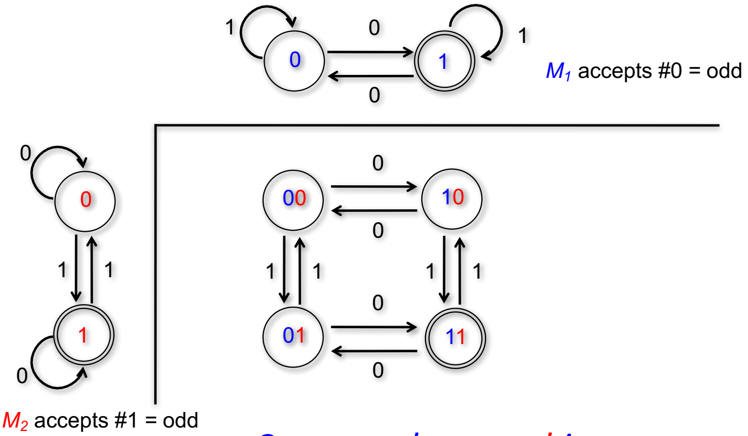
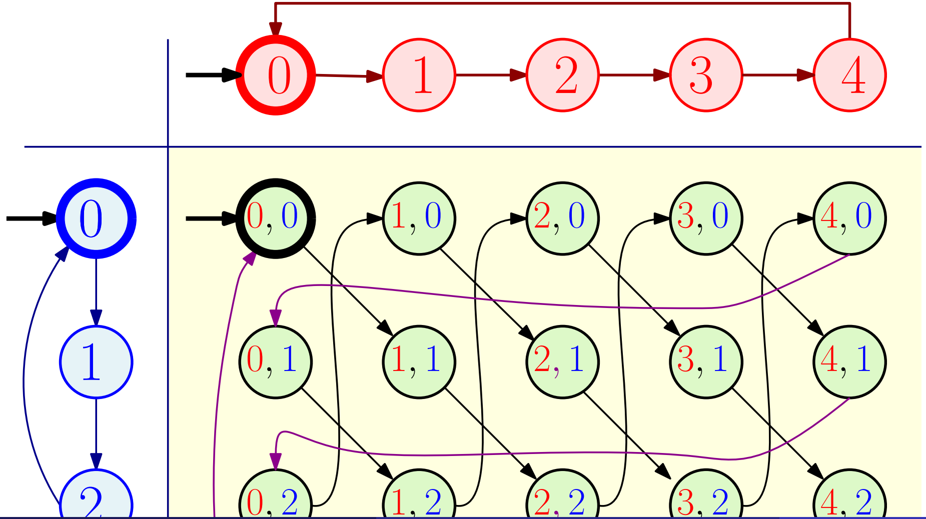


Cross-product machine

Example II

Accept all binary strings of length divisible by 3 and 5



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For each string w, $\delta^*(s, w) = (\delta_1^*(s_1, w), \delta_2^*(s_2, w))$.

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Set Difference

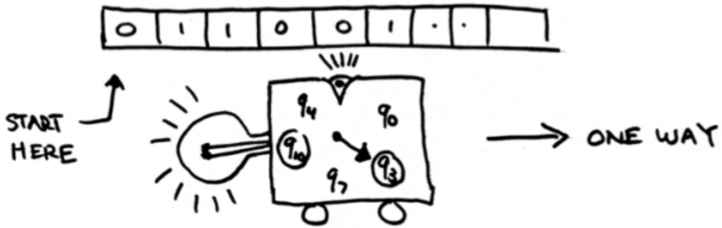
Theorem

M_1, M_2 DFAs. There is a DFA M such that $L(M) = L(M_1) \setminus L(M_2)$.

Exercise: Prove the above using two methods.

- Using a direct product construction
- Using closure under complement and intersection and union

Things to know: 2-way DFA



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Can we allow DFA to scan back and forth? Caveat: Tape is read-only so only memory is in machine's state.

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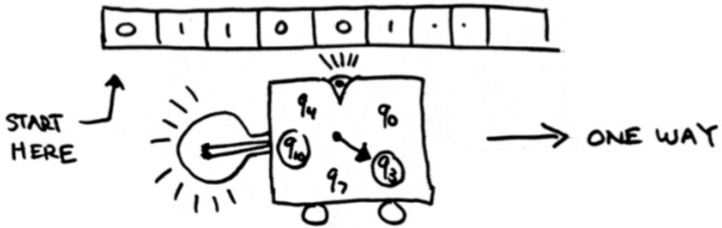

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(for now)