String = finite sequence of symbols

Alphabet Σ

any non-empty finite set

Usually: $\Sigma = \{0,1\}$

Also: $\{\spadesuit \heartsuit \diamondsuit \clubsuit\}$

$\{A\ T\ C\ G\}$

Sequence of X's is

empty or

an X followed by seq of X's

String is either

- empty ε
- symbol followed by string (a, x) or $a \cdot x$ or ax for some $a \in \Sigma$, for some string x

$STRING = (S, (T, (R, (I, (N, (G, \varepsilon))))))$

we'll omit all this syntactic sugar

Length Function

$$|w| = \begin{cases} 0 & \text{if } w = \varepsilon \\ 1 + |x| & \text{if } w = ax \quad \text{for some symbol } a \text{, some string } x \end{cases}$$

$$|STRING| = 1 + |TRING|$$
$$= 2 + |RING| = \cdots = 5 + |G|$$
$$= 6 + |\varepsilon| = 6$$

Concatenation:

$$w \bullet z = \begin{cases} z & \text{if } w = \varepsilon \\ a \cdot (x \bullet z) & \text{if } w = ax \end{cases}$$

function we are defining

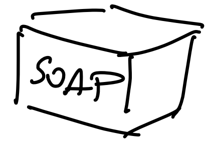
syntactic sugar def string

$$NOW \bullet HERE = N \cdot (OW \bullet HERE)$$
$$= N \cdot (O \cdot (W \bullet HERE))$$
$$= N \cdot (O \cdot (W \cdot (\varepsilon \bullet HERE)))$$
$$= N \cdot (O \cdot (W \cdot HERE))$$
$$= NOWHERE$$

$\neq HERE \bullet NOW$

$= HERENOW$

Theorem: For any string w, we have $w \bullet \varepsilon = w$



Proof: Let w be an arbitrary string

Assume for all string x such that $|x| < |w|$ that $x \bullet \varepsilon = x$. ⟸ "Recursive calls work" "No smaller counterexample"

There are two cases:

- $w = \varepsilon$

$$w \bullet \varepsilon = \varepsilon \bullet \varepsilon \quad \text{[because } w = \varepsilon\text{]}$$
$$= \varepsilon \quad \text{by def of } \bullet$$
$$= w \quad \text{because } w = \varepsilon$$

- $w = ax$

$$w \bullet \varepsilon = ax \bullet \varepsilon \quad [w = ax]$$
$$= a \cdot (x \bullet \varepsilon) \quad \text{[definition of } \bullet\text{]}$$
$$= ax \quad \text{[by ind. hyp!]}$$
$$= w \quad \text{[because } w = ax\text{]}$$

Therefore, in all cases, $w \bullet \varepsilon = w$

Theorem: For all strings w and z, we have $|w \bullet z| = |w| + |z|$

Proof: Let w and z be arbitrary strings

Assume for all string x shorter than w that $|x \bullet z| = |x| + |z|$.

There are two cases:

- case 1: $w = \varepsilon$

$$|w \bullet z| = |\varepsilon \bullet z| \quad [w = \varepsilon]$$
$$= |z| \quad \text{def } \bullet$$
$$= 0 + |z| \quad \text{math}$$
$$= |\varepsilon| + |z| \quad \text{def } |\,|$$
$$= |w| + |z| \quad [w = \varepsilon]$$

- Case 2: $w = ax$

$$|w \bullet z| = |(ax) \bullet z| \quad [w = ax]$$
$$= |a \cdot (x \bullet z)| \quad \text{def } \bullet$$
$$= 1 + \underline{|x \bullet z|} \quad \text{def } |\,|$$

$$= 1 + \underbrace{|x| + |z|} \quad \text{by IH}$$
$$= |ax| + |z| \quad \text{def } |\,|$$
$$= |w| + |z| \quad [w = an]$$

Therefore, in all cases, $|w \cdot z| = |w| + |z|$