Let L be an arbitrary regular language over the alphabet $\Sigma = \{\texttt{0}, \texttt{1}\}$. Prove that the following languages are also regular.

1. $\textsc{Superstrings}(L) := \{xyz \mid y \in L \text{ and } x, z \in \Sigma^*\}$. This language contains all superstrings of strings in L. For example:

$$\textsc{Superstrings}(\{\texttt{10010}\}) = \left\{\underline{\texttt{10010}},\ \texttt{010}\underline{\texttt{10010}},\ \underline{\texttt{10010}}\texttt{11},\ \texttt{100}\underline{\texttt{10010}}\texttt{010},\ \cdots\right\}$$

[Hint: This is much easier than it looks.]

2. $\textsc{Substrings}(L) := \{y \mid x, y, z \in \Sigma^* \text{ and } xyz \in L\}$. This language contains all substrings of strings in L. For example:

$$\textsc{Substrings}(\{\texttt{10010}\}) = \left\{\varepsilon,\ \texttt{0},\ \texttt{1},\ \texttt{00},\ \texttt{01},\ \texttt{10},\ \texttt{001},\ \texttt{010},\ \texttt{100},\ \texttt{0010},\ \texttt{1001},\ \texttt{10010}\right\}$$

3. $\textsc{Cycle}(L) := \{xy \mid x, y \in \Sigma^* \text{ and } yx \in L\}$. This language contains all strings that can be obtained by splitting a string in L into a prefix and a suffix and concatenating them in the wrong order. For example:

$$\textsc{Cycle}(\{\texttt{OOK!},\ \texttt{OOKOOK}\}) = \left\{\texttt{OOK!},\ \texttt{OK!O},\ \texttt{K!OO},\ \texttt{!OOK},\ \texttt{OOKOOK},\ \texttt{OKOOKO},\ \texttt{KOOKOO}\right\}$$

Work on these later.

4. $\textsc{FlipOdds}(L) := \{\mathit{flipOdds}(w) \mid w \in L\}$, where the function *flipOdds* inverts every odd-indexed bit in w. For example:

$$\mathit{flipOdds}(\texttt{\underline{0}0\underline{0}0\underline{1}1\underline{1}1\underline{0}1\underline{0}1\underline{0}1\underline{0}0}) = \texttt{\underline{1}0\underline{1}0\underline{0}1\underline{0}1\underline{1}1\underline{1}1\underline{1}1\underline{1}0}$$

5. $\textsc{UnflipOdd1s}(L) := \{w \in \Sigma^* \mid \mathit{flipOdd1s}(w) \in L\}$, where the function $\mathit{flipOdd1}$ inverts every other $\texttt{1}$ bit of its input string, starting with the first $\texttt{1}$. For example:

$$\mathit{flipOdd1s}(\texttt{0000\underline{1}1\underline{1}100\underline{1}010\underline{1}0}) = \texttt{0000\underline{0}1\underline{0}100\underline{0}010\underline{0}0}$$

6. $\textsc{FlipOdd1s}(L) := \{\mathit{flipOdd1s}(w) \mid w \in L\}$, where the function $\mathit{flipOdd1s}$ is defined in the previous problem.