CS/ECE 374 A ✧ Fall 2023

Practice Final Exam 1

December 5, 2023

Name:	
NetID:	

- ***Don't panic!***
- You have 180 minutes to answer six numbered questions. The questions are described in more detail in a separate handout.
- If you brought anything except your writing implements, your two hand-written double-sided 8½" × 11" cheat sheet, and your university ID, please put it away for the duration of the exam. In particular, please turn off and put away *all* medically unnecessary electronic devices.
- Please clearly print your name and your NetID in the boxes above.
- Please also print your name at the top of every page of the answer booklet, except this cover page. We want to make sure that if a staple falls out, we can reassemble your answer booklet. (It doesn't happen often, but it does happen.)
- **Do not write outside the black boxes on each page**. These indicate the area of the page that our scanner will actually scan. If the scanner can't see your work, we can't grade it.
- If you run out of space for an answer, please use the overflow pages at the back of the answer booklet, but **please clearly indicate where we should look**. If we can't find your work, we can't grade it.
- We will provide scratch paper to anyone who asks, but **only work that is written into the stapled answer booklet will be graded**. In particular, we will not grade any pages that you separate from the answer booklet.
- Proofs or other justifications are required for full credit if and only if we explicitly ask for them, using the word ***prove*** or ***justify*** in bold italics.
- Breathe in. Breathe out. You've got this.

CS/ECE 374 A ✧ Fall 2023 Practice Final 1 Problem 1	Name:

(a) For each statement below, check "YES" if the statement is ***always*** true and "NO" otherwise, and give a ***brief*** (at most one short sentence) explanation of your answer. **Assume** $P \neq NP$**.** If there is any other ambiguity or uncertainty about an answer, write "NO".

- The solution to the recurrence $T(n) = 8T(n/2) + O(n^2)$ is $T(n) = O(n^2)$.

  Yes No ______________________________

- The solution to the recurrence $T(n) = 2T(n/8) + O(n^2)$ is $T(n) = O(n^2)$.

  Yes No ______________________________

- Every directed acyclic graph contains at least one sink.

  Yes No ______________________________

- Given *any* undirected graph $G = (V, E)$, we can compute a spanning tree of G in $O(V + E)$ time using whatever-first search.

  Yes No ______________________________

- Suppose $A[1..n]$ is an array of integers. Consider the following recursive function:

$$What(i,j) = \begin{cases} 0 & \text{if } i < 0 \text{ or } i > n \\ 0 & \text{if } j < 0 \text{ or } j > n \\ \max \begin{Bmatrix} What(i, j-1) \\ What(i-1, j) \\ A[i] \cdot A[j] + What(i+1, j+1) \end{Bmatrix} & \text{otherwise} \end{cases}$$

  We can memoize this function into an array $What[0..n, 0..n]$ in $O(n^2)$ time, by increasing i in the outer loop and increasing j in the inner loop.

  Yes No ______________________________

Problem 1 continues onto the next two pages.

(b) Which of the following statements are true for ***at least one*** language $L \subseteq \{0, 1\}^*$?

- $L^* = (L^*)^*$

  Yes No

- L is decidable, but L^* is undecidable.

  Yes No

- L is neither regular nor NP-hard.

  Yes No

- L is in P, and L has an infinite fooling set.

  Yes No

- The language $\{\langle M \rangle \mid M \text{ accepts } L\}$ is undecidable.

  Yes No

(c) Consider the following pair of languages:

- $\text{DirHamPath} := \left\{ G \;\middle|\; G \text{ is a directed graph with a Hamiltonian path} \right\}$
- $\text{Acyclic} := \left\{ G \;\middle|\; G \text{ is a directed acyclic graph} \right\}$

(For concreteness, assume that in both of these languages, graphs are represented by their adjacency matrices.) Which of the following statements are true, assuming P ≠ NP?

- $\text{Acyclic} \in \text{NP}$

  Yes No

- $\text{Acyclic} \cap \text{DirHamPath} \in \text{P}$

  Yes No

- DirHamPath is decidable.

  Yes No

- A polynomial-time reduction from DirHamPath to Acyclic would imply P=NP.

  Yes No

- A polynomial-time reduction from Acyclic to DirHamPath would imply P=NP.

  Yes No

Problem 1 continues onto the next page.

(d) Suppose there is a ***polynomial-time*** reduction from some language A over the alphabet $\{0, 1\}$ to some other language B over the alphabet $\{0, 1\}$. Which of the following statements are ***always*** true, assuming P $\neq$ NP?

- A is a subset of B.

  Yes No ______________________________

- If $B \in P$, then $A \in P$.

  Yes No ______________________________

- If B is NP-hard, then A is NP-hard.

  Yes No ______________________________

- If B is regular, then A is regular.

  Yes No ______________________________

- If B is regular, then A is decidable.

  Yes No ______________________________

CS/ECE 374 A ✧ Fall 2023 Practice Final 1 Problem 2	Name:

Describe and analyze an algorithm to determine whether the language accepted by a given DFA is finite or infinite. You can assume the input alphabet of the DFA is {0,1}. *[Hint: DFAs are directed graphs.]*

CS/ECE 374 A ✧ Fall 2023 Practice Final 1 Problem 3	Name:

Suppose you are asked to tile a $2 \times n$ grid of squares with dominos (1×2 rectangles). Each domino must cover exactly two grid squares, either horizontally or vertically, and each grid square must be covered by exactly one domino.

Each grid square is worth some number of points, which could be positive, negative, or zero. The ***value*** of a domino tiling is the sum of the points in squares covered by vertical dominos, ***minus*** the sum of the points in squares covered by horizontal dominos.

Describe an algorithm to compute the largest possible value of a domino tiling of a given $2 \times n$ grid. Your input is an array $Points[1\,..\,2, 1\,..\,n]$ of point values.

CS/ECE 374 A ✧ Fall 2023 Practice Final 1 Problem 4	Name:

Submit a solution to *exactly one* of the following problems. Don't forget to tell us which problem you've chosen!

(a) Let Φ be a boolean formula in conjunctive normal form, with exactly three literals per clause (or in other words, an instance of 3SAT). ***Prove*** that it is NP-hard to decide whether Φ has a satisfying assignment in which *exactly half* of the variables are TRUE.

(b) Let $G = (V, E)$ be an arbitrary directed graph whose edges have colors. A *rainbow Hamiltonian cycle* in G is a cycle that visits every vertex of G exactly once, in which no pair of consecutive edges have the same color. ***Prove*** that it is NP-hard to decide whether G has a rainbow Hamiltonian cycle.

(In fact, both of these problems are NP-hard, but we only want a proof for one of them.)

CS/ECE 374 A ✧ Fall 2023 Practice Final 1 Problem 5	Name:

Suppose you are given a height map of a mountain, in the form of an $n \times n$ grid of evenly spaced points, each labeled with an elevation value. You can safely hike directly from any point to any neighbor immediately north, south, east, or west, but only if the elevations of those two points differ by at most Δ. (The value of Δ depends on your hiking experience and your physical condition.)

Describe and analyze an algorithm to determine the longest hike from some point s to some other point t, where the hike consists of an uphill climb (where elevations must increase at each step) followed by a downhill climb (where elevations must decrease at each step). Your input consists of an array $Elevation[1..n, 1..n]$ of elevation values, the starting point s, the target point t, and the parameter Δ.

CS/ECE 374 A ✧ Fall 2023 Practice Final 1 Problem 6	Name:

(a) Let L_a denote the set of all strings in $\{0, 1\}^*$ where every 0 is followed immediately by at least one 1. Describe a DFA or NFA that accepts L_a ***and*** give a regular expression that describes L_a. (You do not need to prove that your answers are correct.)

(b) Let L_b denote the set of all strings in $\{0, 1\}^*$ whose run lengths are increasing; that is, every run except the last is followed immediately by a *longer* run. ***Prove*** that L_b is not a regular language.

(overflow / scratch paper)

(overflow / scratch paper)

(overflow / scratch paper)

(overflow / scratch paper)

(overflow / scratch paper)

Some useful NP-hard problems. You are welcome to use any of these in your own NP-hardness proofs, except of course for the specific problem you are trying to prove NP-hard.

CircuitSat: Given a boolean circuit, are there any input values that make the circuit output True?

3Sat: Given a boolean formula in conjunctive normal form, with exactly three distinct literals per clause, does the formula have a satisfying assignment?

MaxIndependentSet: Given an undirected graph G, what is the size of the largest subset of vertices in G that have no edges among them?

MaxClique: Given an undirected graph G, what is the size of the largest complete subgraph of G?

MinVertexCover: Given an undirected graph G, what is the size of the smallest subset of vertices that touch every edge in G?

MinSetCover: Given a collection of subsets $S_1, S_2, \ldots, S_m$ of a set S, what is the size of the smallest subcollection whose union is S?

MinHittingSet: Given a collection of subsets $S_1, S_2, \ldots, S_m$ of a set S, what is the size of the smallest subset of S that intersects every subset S_i?

3Color: Given an undirected graph G, can its vertices be colored with three colors, so that every edge touches vertices with two different colors?

ChromaticNumber: Given an undirected graph G, what is the minimum number of colors required to color its vertices, so that every edge touches vertices with two different colors?

HamiltonianPath: Given graph G (either directed or undirected), is there a path in G that visits every vertex exactly once?

HamiltonianCycle: Given a graph G (either directed or undirected), is there a cycle in G that visits every vertex exactly once?

TravelingSalesman: Given a graph G (either directed or undirected) with weighted edges, what is the minimum total weight of any Hamiltonian path/cycle in G?

LongestPath: Given a graph G (either directed or undirected, possibly with weighted edges), what is the length of the longest simple path in G?

SteinerTree: Given an undirected graph G with some of the vertices marked, what is the minimum number of edges in a subtree of G that contains every marked vertex?

SubsetSum: Given a set X of positive integers and an integer k, does X have a subset whose elements sum to k?

Partition: Given a set X of positive integers, can X be partitioned into two subsets with the same sum?

3Partition: Given a set X of $3n$ positive integers, can X be partitioned into n three-element subsets, all with the same sum?

IntegerLinearProgramming: Given a matrix $A \in \mathbb{Z}^{n\times d}$ and two vectors $b \in \mathbb{Z}^n$ and $c \in Z^d$, compute $\max\{c \cdot x \mid Ax \leq b, x \geq 0, x \in \mathbb{Z}^d\}$.

FeasibleILP: Given a matrix $A \in \mathbb{Z}^{n\times d}$ and a vector $b \in \mathbb{Z}^n$, determine whether the set of feasible integer points $\max\{x \in \mathbb{Z}^d \mid Ax \leq b, x \geq 0\}$ is empty.

Draughts: Given an $n \times n$ international draughts configuration, what is the largest number of pieces that can (and therefore must) be captured in a single move?

SuperMarioBrothers: Given an $n \times n$ Super Mario Brothers level, can Mario reach the castle?