Tuesday, May 3, 2022 10:54 AM

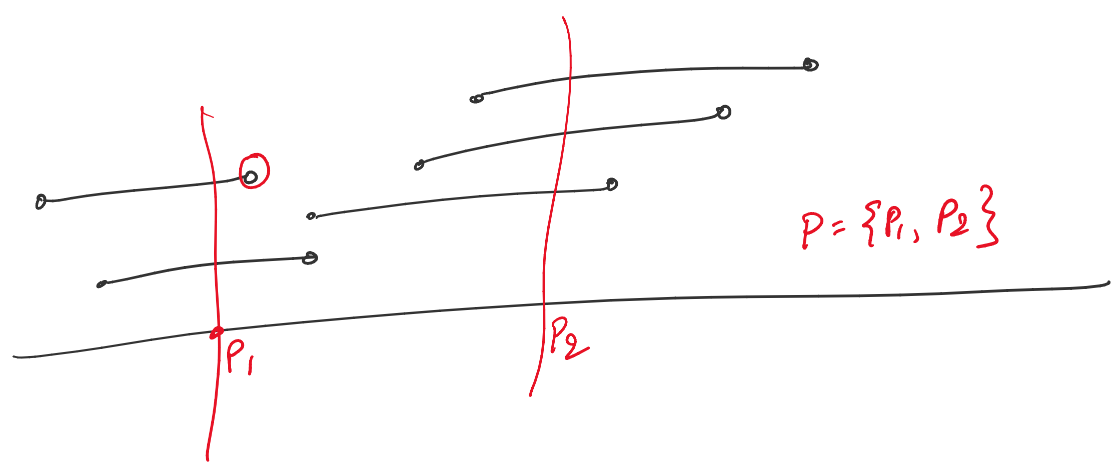
Final Exam: May 12, 8-11 am. Online. Proctored.
Last 30 mins for scan & upload.
So essentially you will have
~150 mins to work on the questions.

Cummulative: Roughly will include
~1 greedy que.
~1 NP-completeness que
short questions on undecidability.

☆ Greedy

MT2 Fodder, #19: Given a set of intervals $[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]$, find the smallest set of points that "stabs" each interval at least once.

eg.



$P = \{P_1, P_2\}$

Greedy Alg'm:

1. $P = \emptyset$, $I = \{1, \ldots, n\}$
2. repeat {
3. pick $i \in I$ w/ smallest b_i
   $i = \text{argmin } b_k$.

2. Rep- ... smallest b_i

$P^G \leftarrow$ 3. Pick $i \in I$ w/ smallest b_i. $i = \arg\min_{k \in I} b_k$.

4. $P = P \cup \{b_i\}$

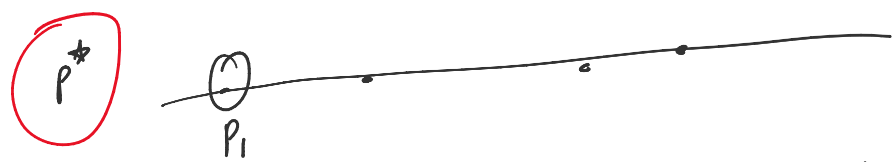
5. Remove all intervals stabbed by b_i from I.

} Until $I = \emptyset$

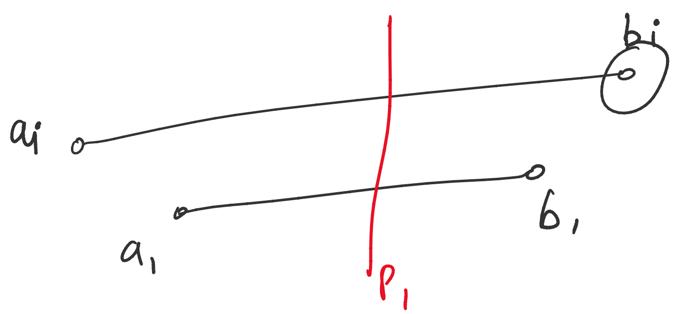
Running time: $O(n \lg n + n)$ like interval scheduling.

Correctness pf: Let P^* be an optimal sol'n.

Let $p_1 \in P^*$ be the leftmost / min point in P^*



Let $b_1 \in P^G$ be the first point picked by greedy.



① $p_1 \le b_1$ o.w. P^* cannot stab $[a_1, b_1]$

② $a_1 \le p_1 \le b_1$ o.w. we can discard p_1 from P^* & still have a feasible sol'n.

Claim: If $[a_i, b_i]$ is stabbed by p_1 then it is also stabbed by b_1

Pf: By choice of $b_1 = \arg\min_{k \in I} b_k$

we know that $a_i \le p \le b_1 \le b_i$

$\Downarrow$

$a_i \leq b_1 \leq b_i$

$\Downarrow$

b_1 stabs $[a_i, b_i]$. ■

$P^* = P^* \setminus \{p_1\} \cup \{b_1\}$. is a feasible & optimal sol'n

Repeat. $\Rightarrow$ By induction $\exists$ an optimal sol'n that exactly matches with the greedy sol'n ■

★ NP-Completeness:

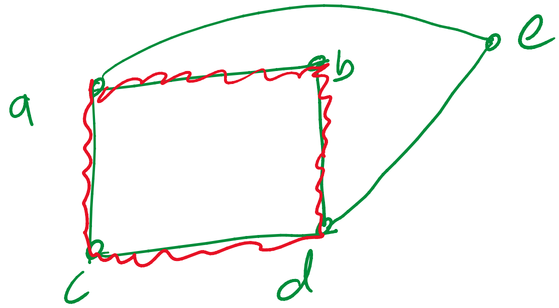
#4. Intra-HC:

input: undirected graph $G=(V,E)$

output: YES iff $\exists$ a closed walk C that visits every vertex exactly once, except one vertex. that it may not visit.

(a).

e.g.



(b) Intra-HC is NP-Complete.

$\swarrow$ O(n) poly-size

... is in NP. ... $v_n, v_1 = C$

(b) Intra- HC is NP-comp

$O(n)$

① Intra-HC is in NP.

certificate : list of vertices $v_1, v_2, \ldots, v_n, v_1 = C$

certifier : ① check if C contains all vertices except one. V_e

Poly-time

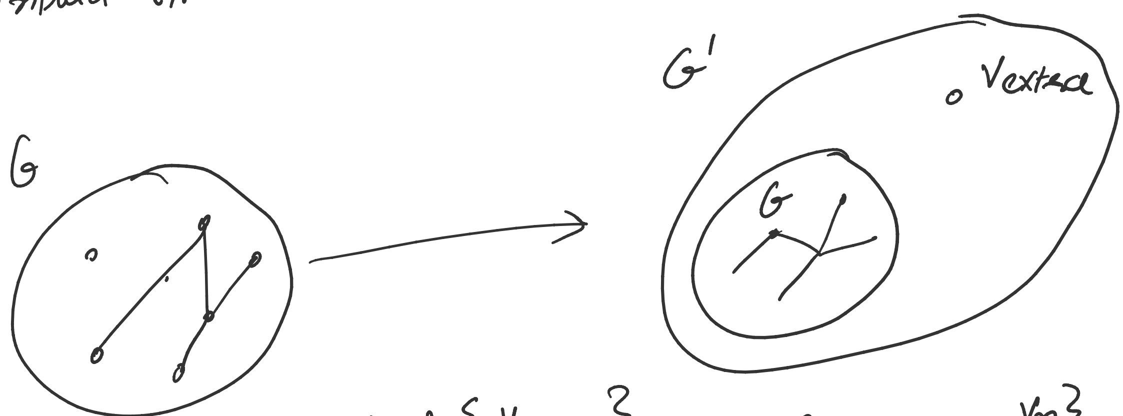
② C is a closed walk

③ every vertex appears exactly once except V_e.

② $HC \leq_p$ Intra- HC.

Given i/p to Ham. Cycle : $G = (V, E)$ undirected.

construct i/p to intra- HC : new graph $G' = (V', E')$ undirected.



$V' = V \cup \{V_{extra}\}$ $\quad V = \{v_1, v_2, \ldots, v_n\}$

$E' = E$

Correctness : $\exists$ HC in $G \iff \exists$ Intra-HC in G'

($\Rightarrow$) let $C = v_1, v_2, \ldots, v_n, v_1$ be a HC in G

then $C' = C$ is an intra- HC in G'

because it covers all vertices of V' exactly once. except V_{extra}

($\Leftarrow$) Let C' be intra- HC in G' ... is

(⇐) Let C' be intra-HC in G' (assuming $|V| \geq 2$) it must be that V_{extra} is missing in C'

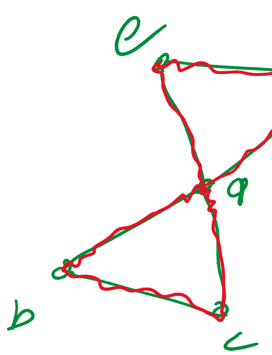
Hence $C = C'$ is a HC in G. ∎

#4 Ultra-HC:

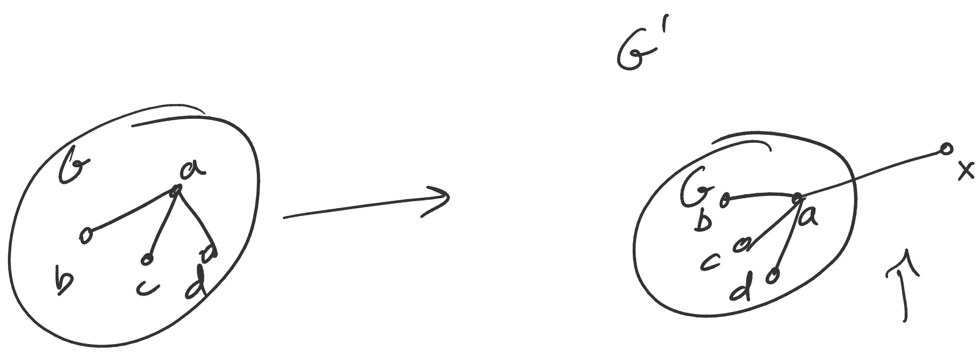
i/p: Graph $G=(V,E)$ undirected.

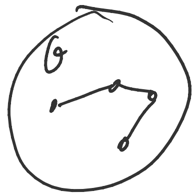
o/p: YES iff $\exists$ a closed walk that contains every vertex exactly once except one vertex that can be repeated.

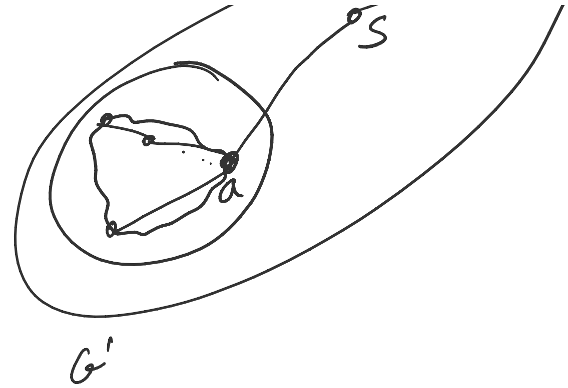
e.g.

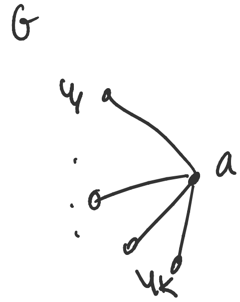


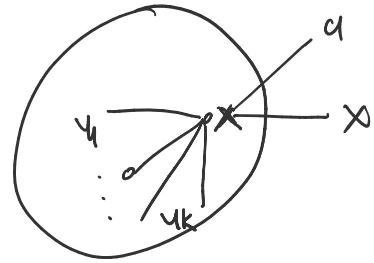
e - d - a - b - c - a - e



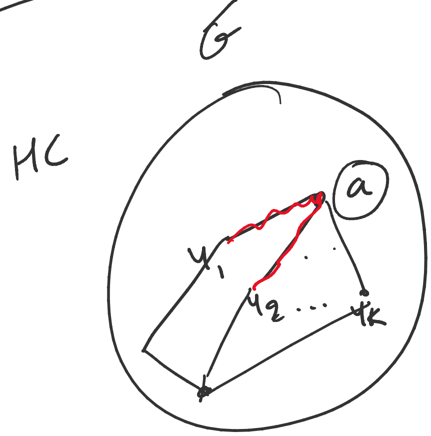


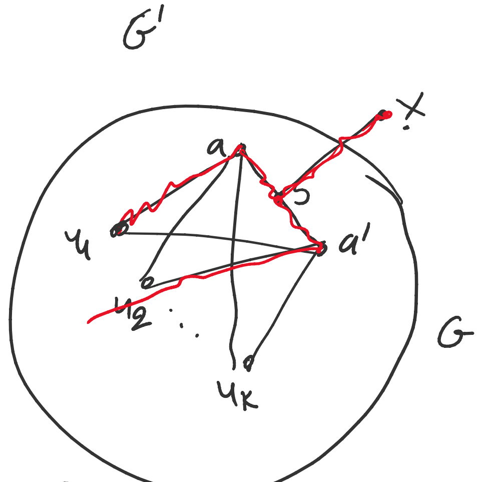






Construction :





$$V' = V \cup \{a', s, x\}$$

$$E' = E \cup \{as, sa', sx\} \cup \{a'u_l \mid u_l a \in E\}$$

Correctness: $\exists$ HC in $G \iff \exists$ Ultra-HC in G'

Correctness: $\exists HC$ in $G \iff \exists$ ultra-HC in G'

($\Rightarrow$) Let C be a HC.

$C = \ldots u_1 - a - u_2 \ldots$

$\ldots u_1 - a - \underline{s} - x - \underline{s} - a' - u_2 \ldots$, is an ultra-HC in G'.

($\Leftarrow$) Let C' be a ultra-HC in G' then s must be repeated in C'

so $C' = \ldots u_l - \underbrace{a - \underline{s} - x - \underline{s} - a'} - u_d \ldots$

$C = \ldots u_l - a - u_d \ldots$

is HC in G. ▲