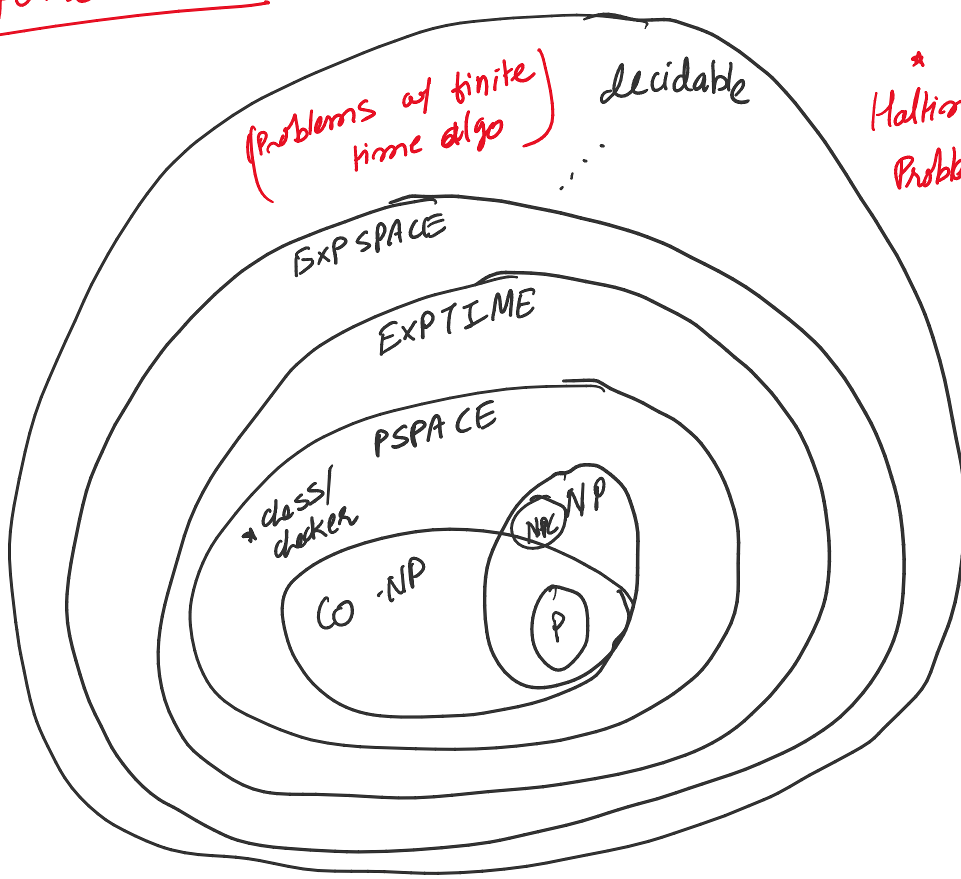
Thursday, April 28, 2022 10:47 AM

Beyond- NP



Un-decidable

* Halting Problem (verification- Style)

EXPTIME $\neq$ P.

PSPACE = NPSPACE.

☆ Un-decidablity:

TM $\equiv$ Alg'm $\equiv$ computer Program.

Given a problem A. $\underline{L(A)} = \{$ input string x $/$ YES on x for A. $\}$

YES $\equiv$ accept.

equivalently NO $\equiv$ reject.

Def: L is decidable if $\exists$ TM/alg'm M that halts on all i/ps.

AND M accepts w iff $w \in L$

M rejects w iff $w \notin L$

☆ Turing's Thm:

★ **Turing's Thm** :
(Halting Prob.)

TM-Halt = $\{ \langle M, w \rangle \mid M \text{ halts on } w \}$

is undecidable.

Similarly.

TM-Acc = $\{ \langle M, w \rangle \mid M \text{ accepts } w \}$

is undecidable.

Pf: By contradiction

suppose TM-ACC is decidable by TM/alg'm Macc

Construct a counter example M_{bad}, w_{bad} :

★ M_{bad} is this alg'm :

on i/p $\langle x \rangle$ $\langle M_{bad} \rangle$, let $M = \langle x \rangle$ (think of x as a description of an alg'm)

" - run Macc on $\langle M, \langle M \rangle \rangle$

- if Macc accepts then reject.
else accept. "

★ $w_{bad} = \langle M_{bad} \rangle$

case I : Macc accepts $\langle M_{bad}, \langle M_{bad} \rangle \rangle$

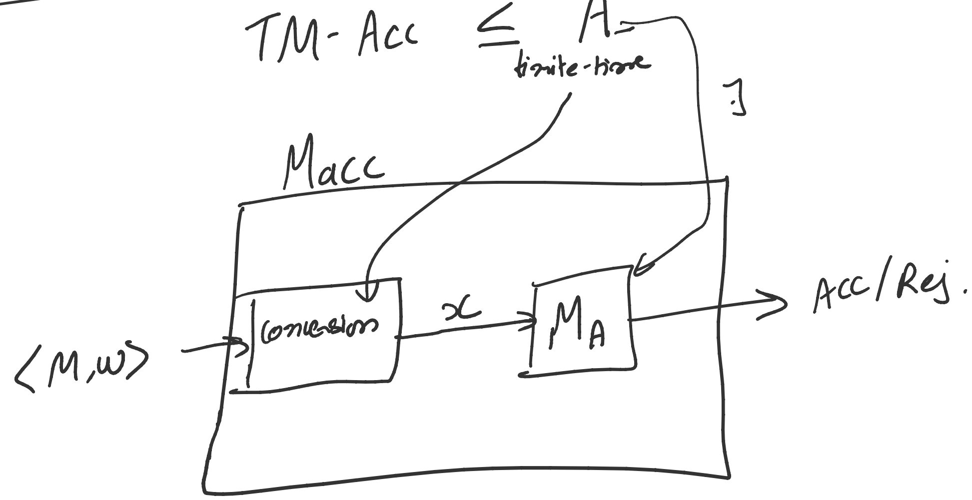
then M_{bad} rejects $\langle M_{bad} \rangle$ wrong!

then M_{bad} rejects $\langle M_{bad} \rangle$ wrong!

Case II: M_{acc} rejects $\langle M_{bad}, \langle M_{bad} \rangle \rangle$

then M_{bad} accepts $\langle M_{bad} \rangle$ wrong! ∎

TM-Acc $\leq_{\text{finite-time}}$ A



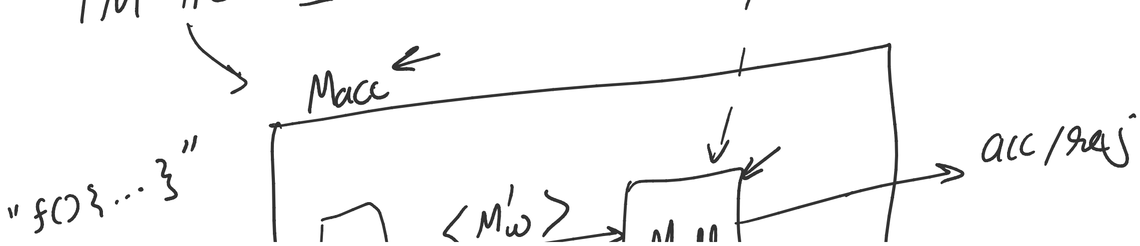
Ex1: TM-ACC-ALL = $\{ \langle M \rangle \mid$ TM M accepts all i/p i.e. $L(M) = \Sigma^* \}$

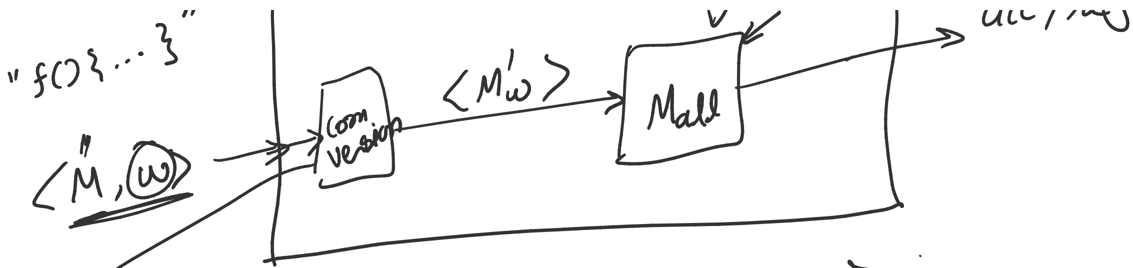
is undecidable.

(Appl'n: main (int n) {
if n is even then accept.
if sum of all divisons of n is 2n then reject
else accept. }

OPEN!

Pf: TM-Acc ≤ TM-Acc-All:





Given i/p to Macc : $\langle M, w \rangle$

Construct i/p to Mall : $\langle M'_w \rangle$ as follows:

$M'_w(x)$: // ignore x, and run M on w.
// M = "f(.){ .... }"

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" f(.) { ....  }
  main (x) {
    if f(w) = accept then return accept
    else          "              reject  }"
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String manipulation

Correctness : TPT Macc accepts $\langle M,w \rangle \iff M$ accepts w.

Macc accepts $\langle M,w \rangle$
$\iff$ Mall accepts $\langle M'_w \rangle$
$\iff$ M'_w accepts all strings
$\iff$ M accepts w.

$\therefore$ Macc decides TM-acc. Contradiction! ∎

Ex 2

TM-acc-some = $\{ \langle M \rangle \mid L(M) \neq \emptyset \}$

is undecidable

same proof as Ex1.

TM-Empty = $\{ \langle M \rangle \mid L(M) = \emptyset \}$

$TM\text{-}Empty = \{ \langle M \rangle \mid L(M) = \emptyset \}$

is undecidable.

(∵ closure under complements)

Ex 3: $TM\text{-}Equiv = \{ \langle M_1, M_2 \rangle \mid L(M_1) = L(M_2) \}$

Pf: $TM\text{-}Acc\text{-}All \leq TM\text{-}Equiv. \rightarrow M_{equiv}$

Given i/p to M_{all} : $\langle M \rangle$

Construct i/p to M_{equiv} : $\langle M, M_2 \rangle$

where M_2 = TM that accept all of Σ^*

= " main(x) { return accept; } "

$\Rightarrow L(M_2) = \Sigma^*$

Correctness:

M_{all} accepts $\langle M \rangle$

$\Leftrightarrow M_{equiv}$ accepts $\langle M, M_2 \rangle$

$\Leftrightarrow L(M) = L(M_2) = \underbrace{\Sigma}_{= \{0,1\}}{}^*$

Ex 4: $TM\text{-}Reg = \{ \langle M \rangle \mid L(M) = \text{regular language} \}$

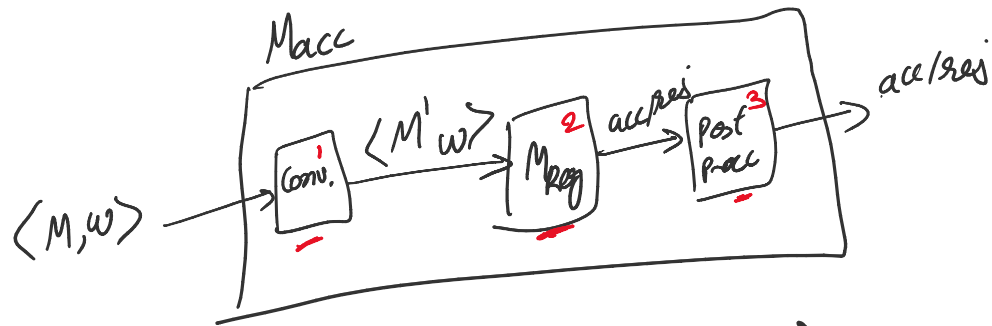
is undecidable.

Pf: Very similar to EX 1. if TM-Reg decidable then M_{Reg} always halts.

$TM\text{-}Acc \leq TM\ Reg \rightarrow$

$M_{...}$

TM-Acc $\leq$



Given i/p to Macc : $\langle M, w \rangle$

Description of Macc machine using MReg machine

1. Construct encoding $M'_w(x)$ as follows:
   - simulate M on w. ← (ignores x)
   - If M accepts w then ← accept x iff x is a palindrom

     Else reject x.

2. Run MReg on $\langle M'_w \rangle$
3. accept iff MReg rejects $\langle M'_w \rangle$

Correctness: M_{acc} accepts $\langle M, w\rangle \Longleftarrow$ accepts w.

Macc accepts $\langle M, w \rangle$

$\Longleftrightarrow$ MReg rejects $\langle M'_w \rangle$

$\Longleftrightarrow$ $L(M'_w) \neq$ regular.

$\Longleftrightarrow$ $L(M'_w) = \{x \mid x \text{ is a palindrom}\}$

$\Longleftrightarrow$ M accepts w.

$(M'_w) = \{x \mid x \text{ is a palindrom}\}$ ← not-regular
if M accepts w

$= \emptyset$ if M rejects w

$= \emptyset$ or M does not halt on w

reg. lang.

Rmk: Proof is very general!

Rice's Thm: Let P be a property about languages.
$\{\langle M\rangle \mid L(M)$ satisfies property $P\}$ is undecidable!

if P is nontrivial
(i.e. $\exists$ a "decidable" lang. with P AND
$\exists$ a " " without P)

Ex 5: $\{\langle M\rangle \mid M$ accepts 374 strings$\}$
is undecidable.

⋮

Rice's Thm not applicable.
$\rightarrow \{\langle M\rangle \mid M$ terminates in 374 steps$\}$
is decidable.

Other Undecidable Problems

a) $\{\langle G\rangle \mid L(G) = \Sigma^*$ for CFG $G\}$
is undecidable

b) Hilbert's 10th thm:
Given a polynomial $P(x_1, \ldots, x_n)$
$\exists$ integer sol'n? is undecidable!

Given a poly... is undecidable!

$\exists$ integer sol'n?

$$x^7+y^7=z^7+10$$