Tuesday, April 26, 2022 10:49 AM

More NP-Completeness :

Last Time : Independent set is NP-Complete

$3SAT \le_p IS$.

Cor: ① VC is NPC ($IS \le_p VC$)

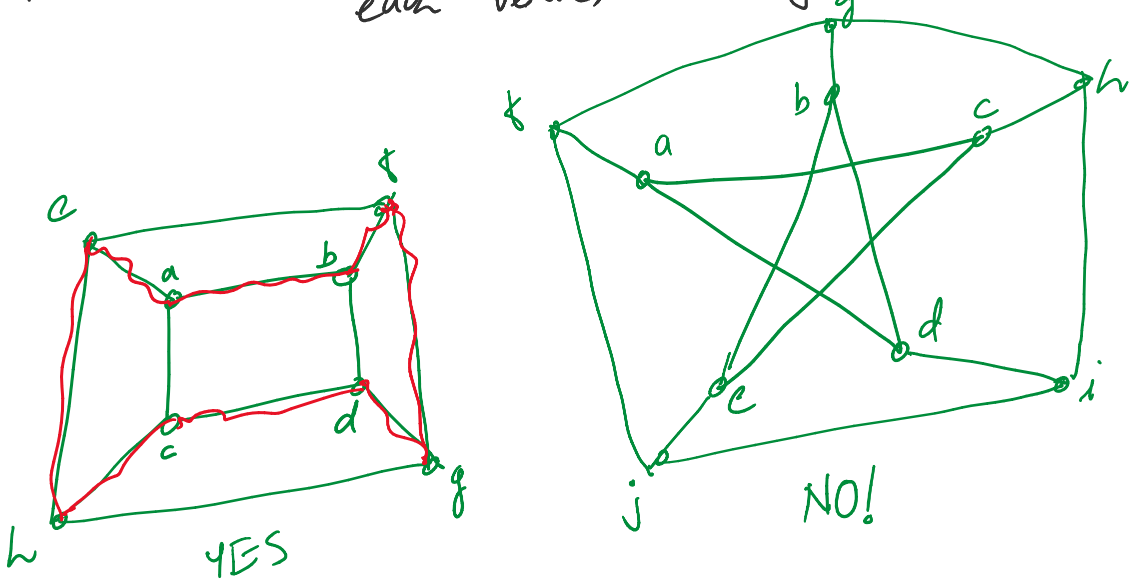
② SC is NPC ($VC \le_p SC$).

☆ Hamiltonian Cycle :

Input: undirected graph $G=(V,E)$

Output: YES iff $\exists$ a cycle that visits each vertex exactly once.

eg.



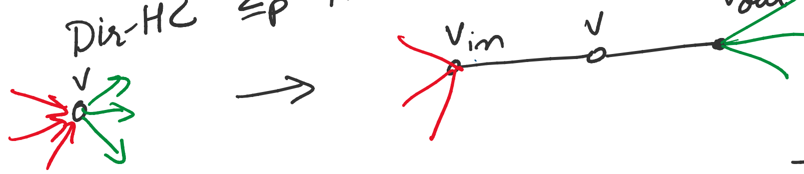
Thm : Dir-HC is NP-Complete

Pf (Karp'75) :

version : HC path (exe)

Dir-HC : Given directed graph G.

Dir-HC $\le_p$ HC.



① Dir-HC is in NP (exe).

② [illegible] $\le_p$ Dir-HC

① Dir-HC is in NP.

② Vertex-Cover $\le_p$ Dir-HC

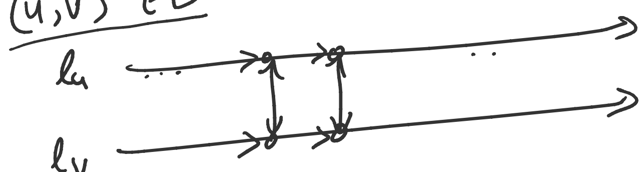
Given an input to vertex cover: undir Graph $G=(V,E)$, int k,

"construct" input to Dir-HC: dir graph G'

as follows: $v \in G$

→ construct line

→ $(u,v) \in E$



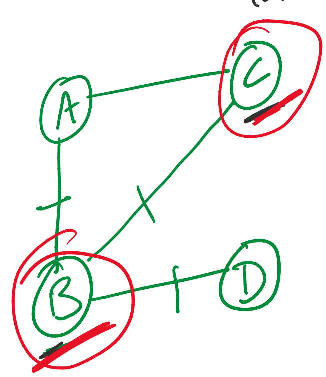
→ add $z_1, \dots, z_k$ vertices, edges to each z_i from end of each l_v, and from each z_i to start of each l_v.

→ # vertices: $k + 4m$, #Edges: $O(nk) + O(m)$

construction is poly-time.

eg.

$G =$

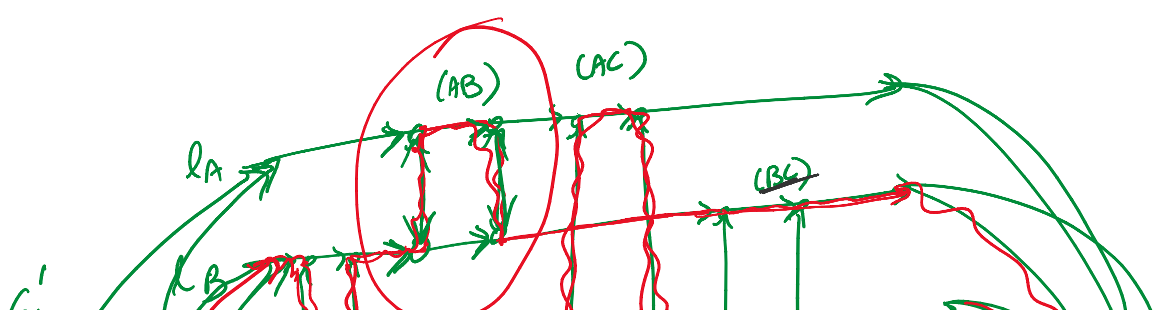


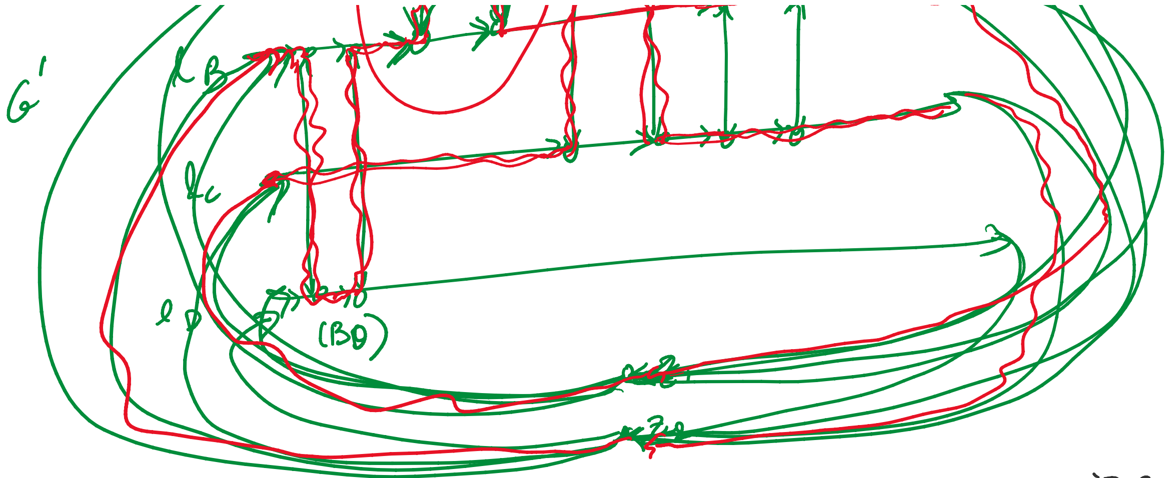
$k=2$

$VC = \{B, C\}$

↓ construction of G'

G'





Correctness: $\exists$ a vertex cover S in G of size $(\leq)$ k.

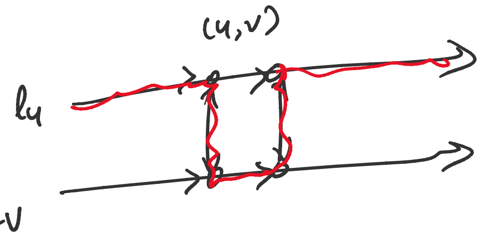
$\iff$ $\exists$ a Ham-Cycle in graph G'.

Pf sketch:

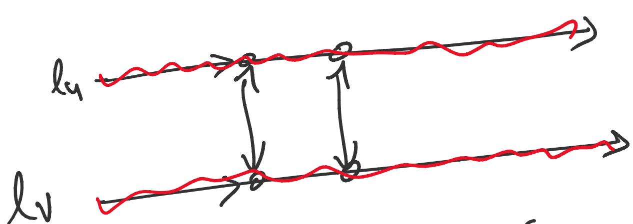
($\Rightarrow$) Let S be a V.C. in G & $|S| = k$.

Then for each $u \in S'$ traverse l_u in cycle C as follows: for each $(u,v) \in E$.

if $v \notin S'$ then



if $v \in S$ then



This constructs k paths. Connect all of them through $z_1, \dots, z_k$ into a single simple cycle.

($\Leftarrow$) Let C be HC in G'.

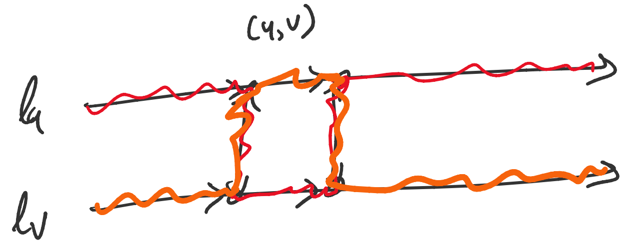
Construct VC S' for G as follows.

$(u,v) \in E$, check how C traverses

Construct VC S ...

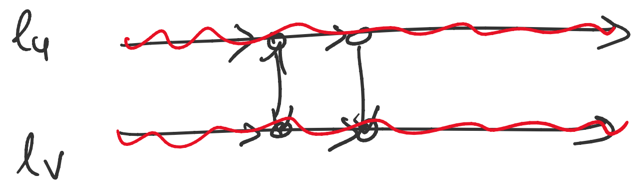
For each $(u,v) \in E$, check how C traverses the corresponding gadget.

C

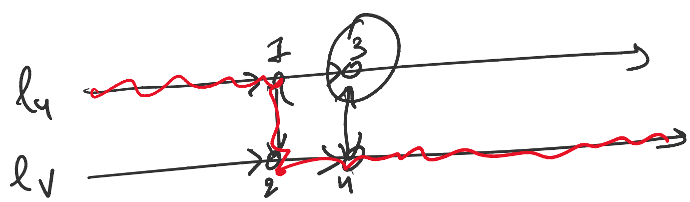


$\Rightarrow$ $u \in S$, $v \notin S$

$u \notin S$, $v \in S$ $\Leftarrow$



$\Rightarrow$ $u \in S$, $v \in S$



NOT POSSIBLE.

It follows that S is a V.C. & can checked that it is consistent □

$|S| = K$ because C covers $z_1 \ldots z_K$ exactly once.

☆ Subset - Sum:

Input: Integers $a_1, a_2, \ldots, a_n, W$ $(0 < a_i \le W)$

Output: YES iff $\exists\, S \subseteq \{a_1, \ldots, a_n\}$ s.t. numbers in S sums to W.

eg. 10, 1, 35, 23, 9 $W = 43$ YES.

Note: Dynamic Prog. $O(n \cdot W)$ time not poly-time. (input size $\sim n \cdot \lg W$) size

Subset sum is NPC.

Thm: Subset sum is NPC.

Pf: ① Subset sum in NP (exe)

② Vertex cover $\le_p$ Subset sum.

Given i/p to VC: Graph $G=(V,E)$, int k

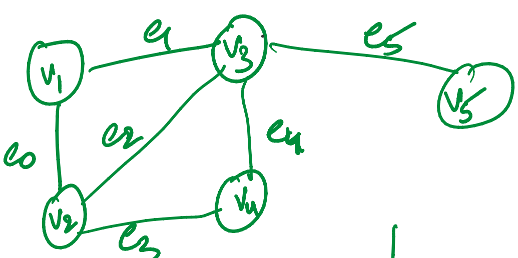
Construct i/p to S.S.: integers $a_1, \dots, a_n, w$

as follows: Let $V = \{v_1, \dots, v_n\}$, $E = \{e_0, \dots, e_{m-1}\}$

$$c_{ij} = \begin{cases} 1 & \text{if } e_j \text{ incident on } v_i \\ 0 & \text{o.w.} \end{cases}$$

$$\left.\begin{aligned} &\forall v_i \in V, \quad a_i = 10^m + \sum_{j=0}^{m-1} c_{ij}\, 10^j \\ &\forall e_j \in E, \quad b_j = 10^j \\ &w = k \times 10^m + \sum_{j=0}^{m-1} 2 \cdot 10^j \end{aligned}\right\} \text{Poly-time.}$$

eg.

$G =$ 

$k=2$

$\{v_2, v_3\}$

↓

(base 10)

incident vertix of G

		e_5	e_4	e_3	e_2	e_1	e_0		
v_1	1	0	0	0	0	1	1	$= a_1$	
* v_2	1	0	0	1	1	0	1	$= a_2$	*
* v_3	1	1	1	0	1	1	0	$= a_3$	*
v_4	1	0	1	1	0	0	0	$= a_4$	
v_5	1	1	0	0	0	0	0	$= a_5$	
							1	$= b_0$	*

v_5	1	1	0	0	0	0	0		
e_0							1	=	b_0 *
e_1						1	0	=	b_1 *
e_2					1	0	0	=	b_2
e_3				1	0	0	0	=	b_3 *
e_4			1	0	0	0	0	=	b_4 *
e_5		1	0	0	0	0	0	=	b_5 *
	K	2	2	2	2	2	2	=	W

Correctness: $\exists$ VC, S in G of size K
$\iff \exists$ subset $S' \in \{a_1, \ldots, a_n, b_0, \ldots, b_{m-1}\}$ that sums to W

($\Rightarrow$) Given V.C. S
construct $S' = \{a_i \mid v_i \in S\} \cup \{b_j \mid e_j \text{ has exactly one incident vertex in } S\}$

Check S' sums to W

($\Leftarrow$) Given S' sums to W
construct V.C. S as follows

$$S = \{v_i \mid a_i \in S'\}$$

$|S| = K$ because leftmost digit in W is K.

& S is a VC of G. because for each edge $e_j = (u,v) \in E$, then a_u or $a_v \in S'$
o.w. j^{th} digit $= 2$ of W cannot be met ∎

o.w. j^{th} digit of

be set ∎

Recap: NP

Cook-Levin → SAT

SAT → 3SAT

3SAT → 3-coloring

3SAT → Independent set ↔ Clique

Vertex cover ↔ Independent set

Vertex cover → H.C.

Vertex cover → subset sum

Vertex cover → set cover