Thursday, April 21, 2022 10:47 AM

Last Time:

NP: (Problems that have "guess-a-sol'n-&-check" type algo)

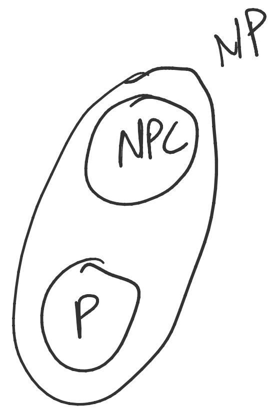
All decision problems of the following form

Input: X

Output: YES iff $\exists y$ s.t. $C(x, y)$ is true.

where ① y (certificate) is poly-size

② C (certifier) is poly-time.



NP-Complete: A problem L is NPC. iff

→ ① $L \in NP$ ← in NP

→ ② $\forall L' \in NP$, $L' \leq_p L$ ← NP-hard

Thm: SAT is NP-Complete.

Fact 5: $\boxed{\text{If ① } L \in NP \Rightarrow L \text{ is in NP. ② } L_0 \leq_p L, \text{ where } L_0 \text{ is NPC. Then } L \text{ is also NP-Complete}}$ ← Recipe to show NPC.

pf: L_0 NPC $\Rightarrow \forall L' \in NP$, $L' \leq_p L_0 \leq_p L$ ($\because$ ②)

($\because$ Fact 2) $\Rightarrow \forall L' \in NP$, $L' \leq_p L$

$\Rightarrow L$ is NP-hard.

Ex1: 3-SAT:

A Boolean formula on n var. of the form

Ex1: 3-SAT.

Input: Boolean formula on n var. of the form

$$F(x_1,\dots,x_n) = \bigwedge_{i=1}^{m} (\ell_{i1} \vee \ell_{i2} \vee \ell_{i3}) \leftarrow \text{clauses}$$

$\uparrow$ literal

3CNF $\leftarrow$ conjunctive normal form

where each ℓ_{ij} is either a var (x_k) or its negation ($\bar{x}_k$)

Output: YES iff $\exists$ an assignment A that makes F true.

eg.

$$F(x_1,x_2,x_3,x_4) = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee \underline{x_2} \vee x_4) \wedge (x_2 \vee \underline{x_3} \vee \bar{x}_4) \wedge (\bar{x}_1 \vee \underline{x_3} \vee \bar{x}_4)$$

A: $(x_1 = x_2 = x_3 = x_4 = 1)$ YES.

Thm: 3SAT is NP-Complete

Pf: (1) $3SAT \in NP$:

certificate: A (n-bits) $\leftarrow$ poly size

certifier: check if F evaluates to true on A.

$\uparrow$ poly time.

(2) $SAT \le_p 3SAT$

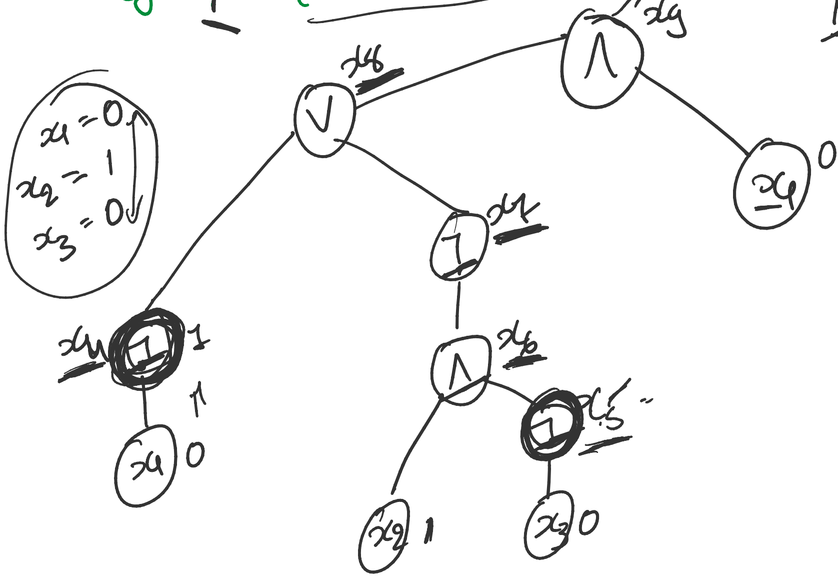
"Given an input to SAT": Boolean formula F

"Construct an input to 3SAT": 3CNF Boolean formula F'

eg. $\underline{F} = (\underline{\bar{x}_1 \vee \overline{x_2 \wedge \bar{x}_3}}) \wedge x_4$ $\quad$ x_5

$\underline{F'} = (x_4 \equiv \bar{x}_1)$ $\quad$ $x_4 = 1$ $\quad$ $x_5 = 1$

eg. $F = (\bar{x}_1 \vee x_2 \wedge x_3) \wedge \ldots$



$$\underline{F'} = (x_4 \equiv \bar{x}_1) \qquad x_4 = \ldots$$
$$\wedge\ (x_5 \equiv \bar{x}_3) \qquad x_5 = 1$$
$$\wedge\ (x_6 \equiv x_2 \wedge x_5) \qquad x_6 = 1$$
$$\wedge\ (x_7 \equiv \bar{x}_6) \qquad x_7 = 0$$
$$\wedge\ (x_8 \equiv x_4 \vee x_7) \qquad x_8 = 1$$
$$\wedge\ (x_9 \equiv x_8 \wedge x_1) \qquad \underline{x_9 = 0}$$
$$\wedge\ x_9$$

$\underline{x_4 \equiv \bar{x}_1}$ $\Rightarrow$ we want $\left.\begin{array}{l}\text{if } x_1 = 0 \text{ then } x_4 = 1 \\ \text{if } x_1 = 1 \text{ then } x_4 = 0\end{array}\right\}$

$$\boxed{(x_1 \vee x_4) \wedge (\bar{x}_1 \vee \bar{x}_4)}$$

(introduce extra vars to make it exactly 3)

$\rightarrow x_6 \equiv x_2 \wedge x_5$

x_2	x_5	x_6
0	0	0
0	1	0
1	0	0
1	1	1

$$(x_2 \vee x_5 \vee \bar{x}_6)$$
$$\wedge\ (x_2 \vee \bar{x}_5 \vee \bar{x}_6)$$
$$\wedge\ (\bar{x}_2 \vee x_5 \vee \bar{x}_6)$$
$$\wedge\ (\bar{x}_2 \vee \bar{x}_5 \vee x_6)$$

$\rightarrow x_8 \equiv x_4 \vee x_7$

x_4	x_7	x_8
0	0	0
0	1	1
1	0	1
1	1	1

$$(x_4 \vee x_7 \vee \bar{x}_8)$$
$$\wedge\ (x_4 \vee \bar{x}_7 \vee x_8)$$
$$\wedge\ (\bar{x}_4 \vee x_7 \vee x_8)$$
$$\wedge\ (\bar{x}_4 \vee \bar{x}_7 \vee x_8)$$

Ex 2: Independent set.

Input: $G=(V,E)$, int k

Output: YES iff $\exists$ an indep-set of size $(\geq) k$

Thm: Independent-set is NP-Complete.

Pf: ① Indep-set $\in$ NP

certificate: $S \subseteq V$ $O(|V|)$ size

certifier: check $|S| \geq k$ & $\forall u,v \in S \Rightarrow uv \notin E$

↳ poly-time.

② $3SAT \leq_p$ Indep-set.

"Given input to 3SAT": 3CNF formula F. on n variables & m clauses.

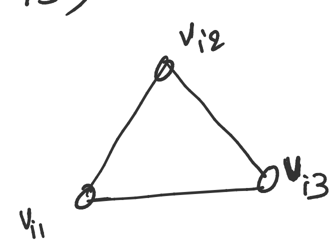
"Construct input to Indep-Set": Graph $G=(V,E)$, int k.

as follows:

① for each clause i: $(\ell_{i1} \vee \ell_{i2} \vee \ell_{i3})$

create 3 vertices v_{i1}, v_{i2}, v_{i3}

& 3 edges $v_{i1}v_{i2}$, $v_{i2}v_{i3}$, $v_{i3}v_{i1}$



② whenever $\ell_{ij} = \overline{\ell_{i'j'}}$, add edge $v_{ij}v_{i'j'}$

③ Set $k=m$.

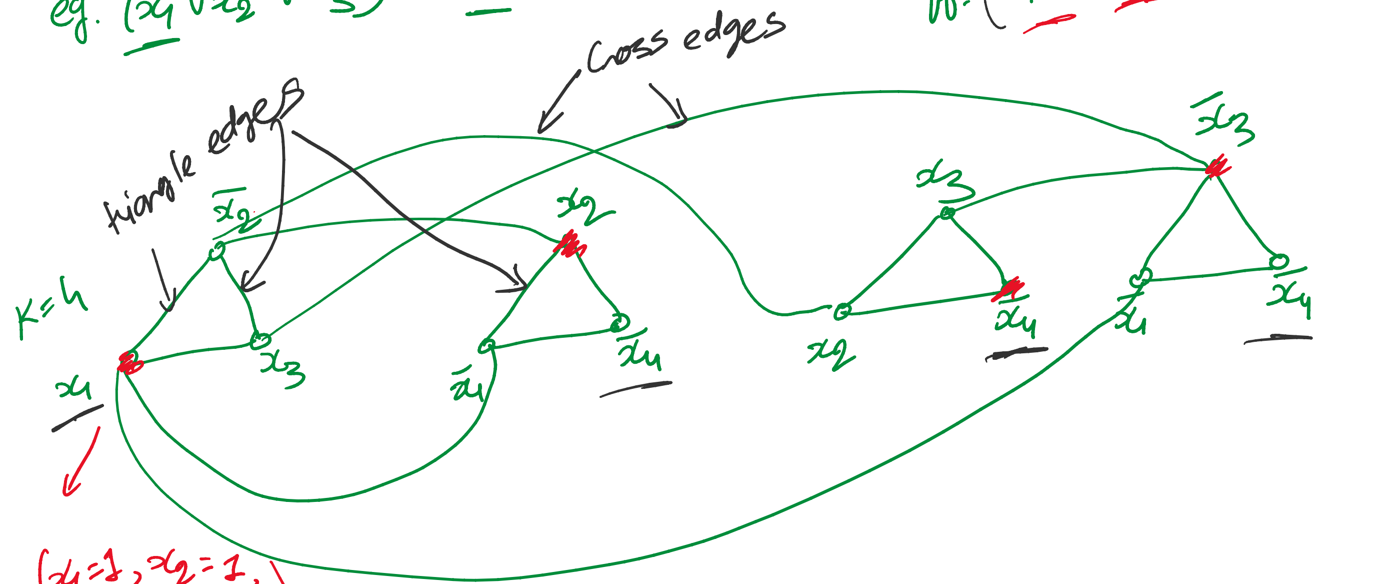
Running Time: $O(m)$ vertices, $O(m^2)$ edges. poly-time.

eg. $(x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_4}) \wedge (x_2 \vee x_3 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee \overline{x_4})$

A: $(x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0)$

... edges

eg. $(x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_4 \vee x_2 \vee x_4) \wedge \ldots$ $\qquad A = (x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0)$



$A = (x_1 = 1, x_2 = 1, x_4 = 0, x_3 = 0)$

* Correctness: $\exists$ assignment A that makes F true

$\iff \exists$ indep-set of size $\geq k$ in graph G.

pf: $(\Rightarrow)$ Let A be a satisfying assignment for F.

Construct subset $S \subseteq V$ as follows:

for each clause $\alpha_{i1} \vee \alpha_{i2} \vee \alpha_{i3}$

pick j s.t. α_{ij} is true in A

put v_{ij} in S.

Then, $|S| = m = k$.

S is an indep-set since

① one vertex from each triangle.
hence no triangle edges bet'n. vertices of S.

② No cross edges either because
at most one vertex from a cross edge in S. given that A is a valid assignment.

($\Leftarrow$) Let S be an indep-set of size $\geq k = m$ in graph G.

Construct assignment A as follows:

whenever $v_{ij} \in S$, set ℓ_{ij} to true in A

$$\left[\begin{array}{l} \text{if } \ell_{ij} = x_k \text{ then set } x_k = 1 \\ \text{if } \ell_{ij} = \overline{x_k} \text{ then set } x_k = 0 \end{array}\right)$$

(set all other variables arbitrarily)

Claim 1: A is consistent.

Pf: If $\ell_{ij} = \overline{\ell_{i'j'}}$ then $v_{ij} v_{i'j'} \in E$

$\Rightarrow$ only one of v_{ij} & $v_{i'j'}$ in S

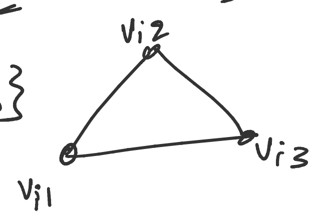
$\Rightarrow$ we "do not" set both ℓ_{ij} & $\ell_{i'j'}$ to true.

$\Rightarrow$ A is consistent.

Claim 2: A is a satisfying assignment for F.

Pf: For each clause i, $(\ell_{i1} \vee \ell_{i2} \vee \ell_{i3})$

$\therefore$ atmost 1 of $\{v_{i1}, v_{i2}, v_{i3}\}$ is in S.



But since $|S| \geq k = m$, exactly one of $v_{ij} \in S \Rightarrow \ell_{ij}$ is true for some $j = 1, 2, 3$

$\Rightarrow$ clause i is true as per A.

$\Rightarrow$ clause i is true as per v.

This holds for every clause i

$\Rightarrow$ A is satisfying assignment for F ∎