Prove that each of the following languages is ***not*** regular.

1 $\{0^{2^n} \mid n \geq 0\}$.

Solution:

Choose $F = \{0^{2^n} \mid n \geq 0\}$.

Let x and y be two arbitrary strings of F with $x \neq y$.

Then $x = 0^{2^i}$ and $y = 0^{2^j}$ for some non-negative integers $i \neq j$.

Choose $z = 0^{2^i}$.

Then $xz = 0^{2^i}0^{2^i} = 0^{2^{i+1}} \in L$.

And $yz = 0^{2^j}0^{2^i} = 0^{2^i+2^j} \notin L$, because $i \neq j$ (since $2^i + 2^j$ cannot be a power of 2).

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular.

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2 $\{0^{2n}1^n \mid n \geq 0\}$

Solution:

Choose $F = \{0^i \mid i \geq 0\}$.

Let x and y be two arbitrary strings in F with $x \neq y$.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Choose $z = 0^i1^i$.

Then $xz = 0^{2i}1^i \in L$.

And $yz = 0^{i+j}1^i \notin L$, because $i + j \neq 2i$.

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular.

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3 $\{0^m1^n \mid m \neq 2n\}$

Solution:

Choose $F = \{0^i \mid i \geq 0\}$.

Let x and y be two arbitrary strings in F with $x \neq y$.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Choose $z = 0^i1^i$.

Then $xz = 0^{2i}1^i \notin L$.

And $yz = 0^{i+j}1^i \in L$, because $i + j \neq 2i$.

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular.

4 Strings over $\{0, 1\}$ where the number of 0s is exactly twice the number of 1s.

Solution:

Choose $F = \{0^i \mid i \geq 0\}$.

Let x and y be two arbitrary strings in F with $x \neq y$.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Choose $z = 0^i 1^i$.

Then $xz = 0^{2i} 1^i \in L$.

And $yz = 0^{i+j} 1^i \notin L$, because $i + j \neq 2i$.

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular.

Solution:

If L were regular, then the language

$$\big((0+1)^* \setminus L\big) \cap 0^*1^* \;=\; \{0^m 1^n \mid m \neq 2n\}$$

would also be regular, because regular languages are closed under complement and intersection. But we just proved that $\{0^m 1^n \mid m \neq 2n\}$ is not regular in problem 3. *[This proof would be worth full credit in homework or an exam, if we do not explicitly specify that you should use the fooling set method.]*

5 Strings of properly nested parentheses (), brackets [], and braces {}. For example, the string ([]){} is in this language, but the string ([ )] is not, because the left and right delimiters don't match.

Solution:

Choose $F = \{(^i \mid i \geq 0\}$.

Let x and y be two arbitrary strings in F with $x \neq y$.

Then $x = (^i$ and $y = (^j$ for some non-negative integers $i \neq j$.

Choose $z =)^i$.

Then $xz = (^i)^i \in L$.

And $yz = (^j)^i \notin L$, because $i \neq j$.

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular.

6 Strings of the form $w_1\#w_2\#\cdots\#w_n$ for some $n \geq 2$, where each substring w_i is a string in $\{0, 1\}^*$, and some pair of substrings w_i and w_j are equal.

Solution:

Choose $F = \{0^i \mid i \geq 0\}$.

Let x and y be arbitrary strings in F with $x \neq y$.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Choose $z = \#0^i$.

Then $xz = 0^i\#0^i \in L$.

And $yz = 0^j\#0^i \notin L$, because $i \neq j$.

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular.

Extra problems

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7 $\{w \in (0+1)^* \mid w$ is the binary representation of a perfect square$\}$

Solution:

Idea: We design our fooling set around numbers of the form $(2^k+1)^2 = 2^{2k}+2^{k+1}+1$, which has binary representation $10^{k-2}10^k1$. The argument is somewhat simpler if we further restrict k to be even.

Choose $F = \{10^{2i}1 \mid i \geq 0\}$.

Let x and y be two distinct arbitrary strings in F.

Then $x = 10^{2i-2}1$ and $y = 10^{2j-2}1$, for some positive integers $i \neq j$. Without loss of generality, assume $i < j$. (Otherwise, swap x and y.)

Choose $z = 0^{2i}1$.

Then $xz = 10^{2i-2}10^{2i}1$ is the binary representation of $2^{4i} + 2^{2i+1} + 1 = (2^{2i}+1)^2$, and therefore $xz \in L$.

On the other hand, $yz = 10^{2j-2}10^{2i}1$ is the binary representation of $2^{2i+2j} + 2^{2i+1} + 1$. Simple algebra gives us the inequalities

$$\begin{aligned}
(2^{i+j})^2 &= 2^{2i+2j} \\
&< \mathbf{2^{2i+2j} + 2^{2i+1} + 1} \\
&< 2^{2(i+j)} + 2^{i+j+1} + 1 \\
&= (2^{i+j}+1)^2.
\end{aligned}$$

So $2^{2i+2j} + 2^{2i+1} + 1$ lies between two consecutive perfect squares, and thus is not a perfect square, which implies that $yz \notin L$.

We conclude that F is a fooling set for L. Because F is infinite, L cannot be regular.