CS/ECE 374 A (Spring 2022)
Past HW1 Problems with Solutions

Problem Old.1.1: Let $L \subseteq \{0,1\}^*$ be a language defined recursively as follows:

- $\varepsilon \in L$.
- For all $w \in L$ we have $0w1 \in L$.
- For all $x, y \in L$ we have $xy \in L$.
- And these are all the strings that are in L.

Prove, by induction, that for any $w \in L$, and any prefix u of w, we have that $\#_0(u) \geq \#_1(u)$. Here $\#_0(u)$ is the number of 0 appearing in u ($\#_1(u)$ is defined similarly). You can use without proof that $\#_0(xy) = \#_0(x) + \#_0(y)$, for any strings x, y.

Solution:

Proof. The proof is by induction on the length of w.

Base case: If $|w| = 0$ then $w = \varepsilon$, and then $\#_0(w) = 0 \geq \#_1(u) = 0$. Since the only prefix of the empty string is itself, the claim readily follows.

Induction hypothesis: Assume that the claim holds for all strings of length $< n$.

Induction step: We need to prove the claim for a string w of length n. There are two possibilities:

- $w = 0z1$, for some string $z \in L$.
  Let u be any prefix of w. If $u = \varepsilon$ or $u = 0$ then the claim clearly holds for u.
  If $u = w$, then
  $$\#_0(u) = \#_0(w) = 1 + \#_0(z) + 0 \geq 1 + \#_1(z) = \#_1(w) = \#_1(u),$$
  which implies the claim (we used the induction hypothesis on z, since $z \in L$ and $|z| = |w| - 2 < n$).
  So the remaining case is when $u = 0z'$, where z' is a prefix of z. In this case,
  $$\#_0(u) = \#_0(0z') = 1 + \#_0(z') \geq 1 + \#_1(z') = 1 + \#_1(u) > \#_1(u),$$
  Again, we used the induction hypothesis on z, since $z \in L$, z' is a prefix of z, and z strictly shorter than w. This implies the claim.
- $w = xy$, for some strings $x, y \in L$, such that $|x|, |y| > 0$.
  Let u be a prefix of w. If u is a prefix of x, then the claim holds readily by induction. The remaining case is when $u = xz$, for some z which is prefix of y. Here,
  $$\#_0(u) = \#_0(xz) = \#_0(x) + \#_0(z) \geq \#_1(x) + \#_1(z) = \#_1(u),$$
  by using the induction hypothesis on x (which is a prefix of itself), and on z (which is a prefix of y), noting that both x and y are strictly shorter than w.

$\square$

Problem Old.1.2: Consider the recurrence

$$T(n) = \begin{cases} T(\lfloor n/3 \rfloor) + T(\lfloor n/4 \rfloor) + T(\lfloor n/5 \rfloor) + T(\lfloor n/6 \rfloor) + n & n \geq 6 \\ 1 & n < 6. \end{cases}$$

Prove by induction that $T(n) = O(n)$.

Solution:

Claim 1. *For $c \geq 20$, and for all $n \geq 1$, we have $T(n) \leq cn$.*

Proof. **Base case.** For $n < 6$ the claim holds for any $c \geq 1$ by definition.

Induction hypothesis. Let $n \geq 6$. Assume that $T(k) \leq ck$ for all $1 \leq k < n$.

Induction step. We need to prove that $T(n) \leq cn$. We know that

$$\begin{aligned} T(n) &= T(\lfloor n/3 \rfloor) + T(\lfloor n/4 \rfloor) + T(\lfloor n/5 \rfloor) + T(\lfloor n/6 \rfloor) + n \\ &\leq c\lfloor n/3 \rfloor + c\lfloor n/4 \rfloor) + c\lfloor n/5 \rfloor) + c\lfloor n/6 \rfloor) + n \qquad \text{(by the induction hypothesis)} \\ &\leq cn/3 + cn/4 + cn/5 + cn/6 + n \\ &\leq \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right)cn + n \;=\; \left(\frac{3}{4} + \frac{1}{5}\right)cn + n \;=\; \left(\frac{19}{20}c + 1\right)n \;\leq\; cn, \end{aligned}$$

provided that

$$\frac{19}{20}c + 1 \leq c \iff 1 \leq \frac{1}{20}c \iff c \geq 20.$$

$\square$

IMPORTANT NOTE: make sure that the "c" in the conclusion from the induction step ($T(n) \leq cn$) is the same as the "c" you start with from the induction hypothesis ($T(k) \leq ck$ for $k < n$). If not (for example, if you could only conclude that $T(n) \leq 1.01cn$), then the whole proof would be incorrect—because the constant factor will "blow up" when we repeat!

(General advice: avoid big-O notation inside induction proofs!)