

# Greedy Algorithms

## Lecture 19

Wednesday, April 1, 2020

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# Part I

## Greedy Algorithms: Tools and Techniques

# What is a Greedy Algorithm?

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Greedy algorithms:

- 1 make decision incrementally in small steps *without backtracking*
- 2 decision at each step is based on improving *local or current* state in a myopic fashion without paying attention to the *global* situation
- 3 decisions often based on some fixed and simple *priority* rules

# Pros and Cons of Greedy Algorithms

## Pros:

- 1 Usually (too) easy to design greedy algorithms
- 2 Easy to implement and often run fast since they are simple
- 3 Several important cases where they are effective/optimal
- 4 Lead to a first-cut heuristic when problem not well understood

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- 2 Many greedy algorithms possible for a problem and no structured way to find effective ones

CS 374: Every greedy algorithm needs a proof of correctness



# Greedy Algorithm Types

Crude classification:

- 1 **Non-adaptive:** fix some ordering of decisions a priori and stick with the order
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Plan:

- 1 See several examples
- 2 Pick up some proof techniques

## Part II

# Scheduling Jobs to Minimize Average Waiting Time

# The Problem

- $n$  jobs  $J_1, J_2, \dots, J_n$ .  $J_i$  has non-negative processing time  $p_i$
- One server/machine/person available to process jobs.
- Schedule/order jobs to min. total or average *waiting time*
- Waiting time of  $J_i$  in schedule  $\sigma$ : sum of processing times of all jobs scheduled before  $J_i$

	$J_1$	$J_2$	$J_3$	$J_4$	$J_5$	$J_6$
<i>time</i>	3	4	1	8	2	6

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**Optimal schedule:**

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**Optimal schedule:** Shortest Job First.  $J_3, J_5, J_1, J_2, J_6, J_4$ .

# Optimality of Shortest Job First (SJF)

## Theorem

*Shortest Job First gives an optimum schedule for the problem of minimizing total waiting time.*



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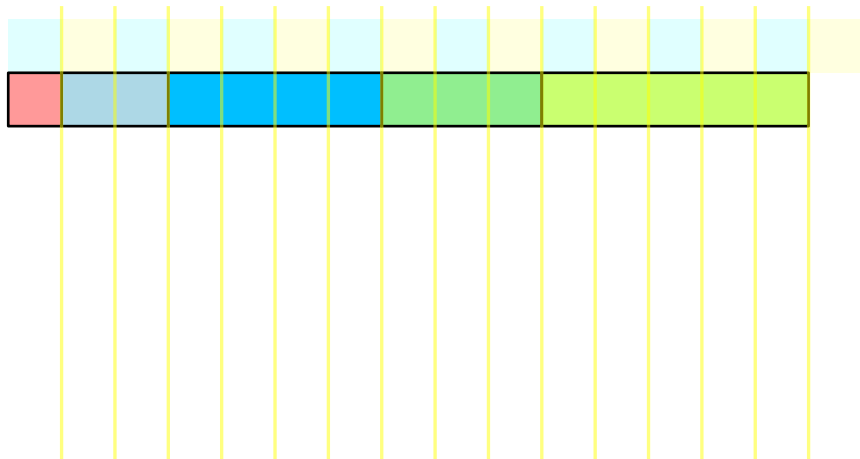
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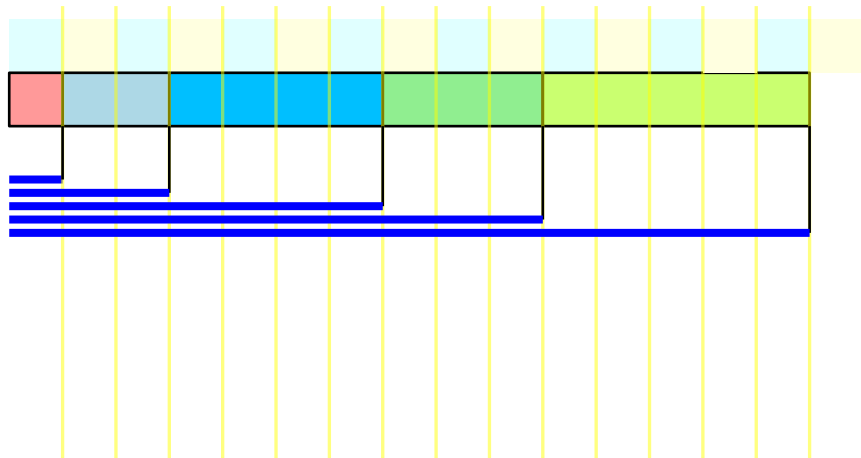
**Proof strategy:** exchange argument

Assume without loss of generality that job sorted in increasing order of processing time and hence  $p_1 \leq p_2 \leq \dots \leq p_n$  and SJF order is  $J_1, J_2, \dots, J_n$ .

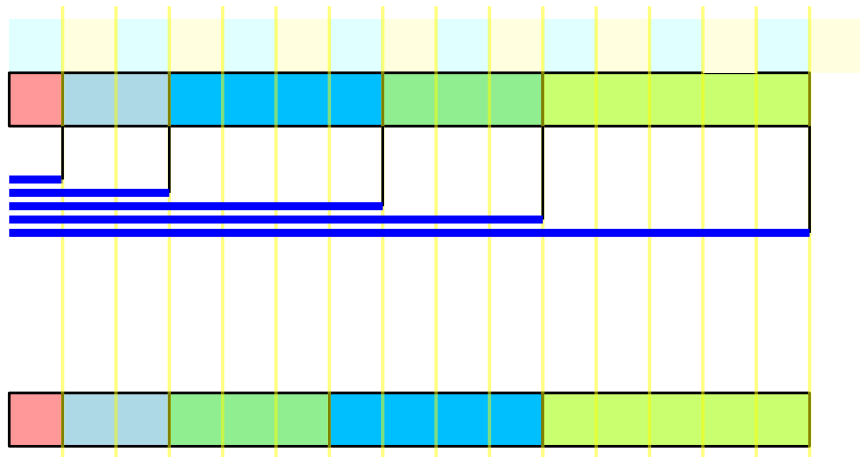
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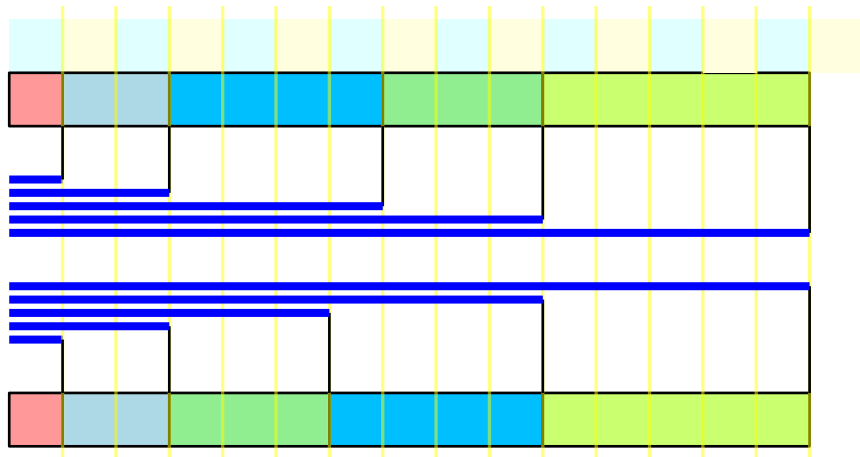
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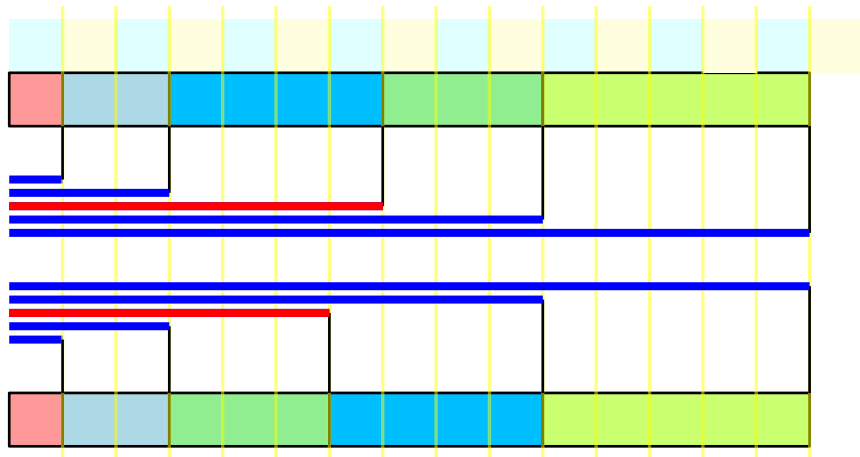
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# Inversions

## Definition

A schedule  $J_{i_1}, J_{i_2}, \dots, J_{i_n}$  has an **inversion** if there are jobs  $J_a$  and  $J_b$  such that  $S$  schedules  $J_a$  before  $J_b$ , but  $p_a > p_b$ .



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## Claim

*If a schedule has an inversion then there is an inversion between two adjacently scheduled jobs.*

Proof: exercise.

# Proof of optimality of SJF

SJF = Shortest Job First

Recall **SJF** order is  $J_1, J_2, \dots, J_n$ .

- Let  $J_{i_1}, J_{i_2}, \dots, J_{i_n}$  be an optimum schedule with fewest inversions.
- If schedule has no inversions then it is identical to **SJF** schedule and we are done.
- Otherwise there is an  $1 \leq \ell < n$  such that  $i_\ell > i_{\ell+1}$  since schedule has inversion among two adjacently scheduled jobs

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## Claim

*The schedule obtained from  $J_{i_1}, J_{i_2}, \dots, J_{i_n}$  by exchanging/swapping positions of jobs  $J_{i_\ell}$  and  $J_{i_{\ell+1}}$  is also optimal and has one fewer inversion.*

Assuming claim we obtain a contradiction and hence optimum schedule with fewest inversions must be the **SJF** schedule.

# A Weighted Version

- $n$  jobs  $J_1, J_2, \dots, J_n$ .  $J_i$  has non-negative processing time  $p_i$  and a non-negative weight  $w_i$
- One server/machine/person available to process jobs.
- Schedule/order the jobs to minimize total or average *waiting time*
- Waiting time of  $J_i$  in schedule  $\sigma$ : sum of processing times of all jobs scheduled before  $J_i$
- Goal: minimize total *weighted* waiting time.

	$J_1$	$J_2$	$J_3$	$J_4$	$J_5$	$J_6$
<i>time</i>	3	4	1	8	2	6
<i>weight</i>	10	5	2	100	1	1

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## Part III

# Scheduling to Minimize Lateness

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- ① Given jobs  $J_1, J_2, \dots, J_n$  with deadlines and processing times to be scheduled on a single resource.
- ② If a job  $i$  starts at time  $s_i$  then it will finish at time  $f_i = s_i + t_i$ , where  $t_i$  is its processing time.  $d_i$ : deadline.
- ③ The lateness of a job is  $\ell_i = \max(0, f_i - d_i)$ .
- ④ Schedule all jobs such that  $L = \max \ell_i$  is **minimized**.

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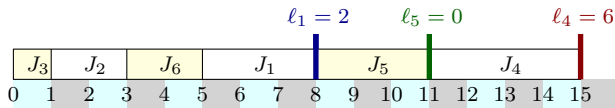
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$t_i$	3	2	1	4	3	2
$d_i$	6	8	9	9	14	15



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# Greedy Template

Initially  $R$  is the set of all requests

$curr\_time = 0$

$max\_lateness = 0$

**while**  $R$  is not empty **do**

    choose  $i \in R$

$curr\_time = curr\_time + t_i$

**if** ( $curr\_time > d_i$ ) **then**

$max\_lateness = \max(curr\_time - d_i, max\_lateness)$

**return**  $max\_lateness$

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         $max\_lateness = \max(curr\_time - d_i, max\_lateness)$   
  
return  $max\_lateness$ 
```

**Main task:** Decide the order in which to process jobs in  $R$

# Three Algorithms

- ① Shortest job first — sort according to  $t_i$ .
- ② Shortest slack first — sort according to  $d_i - t_i$ .
- ③ **EDF** = Earliest deadline first — sort according to  $d_i$ .

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Counter examples for first two: exercise

# Earliest Deadline First

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Proof via an exchange argument.

Idle time: time during which machine is not working.

## Lemma

*If there is a feasible schedule then there is one with no idle time before all jobs are finished.*

# Inversions

EDF = Earliest Deadline First

Assume jobs are sorted such that  $d_1 \leq d_2 \leq \dots \leq d_n$ . Hence EDF schedules them in this order.

## Definition

A schedule  $S$  is said to have an **inversion** if there are jobs  $i$  and  $j$  such that  $S$  schedules  $i$  before  $j$ , but  $d_i > d_j$ .

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## Claim

*If a schedule  $S$  has an inversion then there is an inversion between two adjacently scheduled jobs.*

Proof: exercise.

# Proof sketch of Optimality of EDF

- Let  $S$  be an optimum schedule with smallest number of inversions.
- If  $S$  has no inversions then this is same as EDF and we are done.
- Else  $S$  has two adjacent jobs  $i$  and  $j$  with  $d_i > d_j$ .
- Swap positions of  $i$  and  $j$  to obtain a new schedule  $S'$

## Claim

*Maximum lateness of  $S'$  is no more than that of  $S$ . And  $S'$  has strictly fewer inversions than  $S$ .*

# Proof sketch of Optimality of EDF

## Part IV

# Maximum Weight Subset of Elements: Cardinality and Beyond

# Picking $k$ elements to maximize total weight

- 1 Given  $n$  items each with non-negative weights/profits and integer  $1 \leq k \leq n$ .
- 2 Goal: pick  $k$  elements to **maximize** total weight of items picked.

	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$
<i>weight</i>	3	2	1	4	3	2

$k = 2$ :

$k = 3$ :

$k = 4$ :

# Greedy Template

```
N is the set of all elements  $X \leftarrow \emptyset$   
(*  $X$  will store all the elements that will be picked *)  
while  $|X| < k$  and N is not empty do  
    choose  $e_j \in N$  of maximum weight  
    add  $e_j$  to  $X$   
    remove  $e_j$  from N  
return the set  $X$ 
```

**Remark:** One can rephrase algorithm simply as sorting elements in decreasing weight order and picking the top  $k$  elements but the above template generalizes to other settings a bit more easily.



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```

**Remark:** One can rephrase algorithm simply as sorting elements in decreasing weight order and picking the top  $k$  elements but the above template generalizes to other settings a bit more easily.

## Theorem

*Greedy is optimal for picking  $k$  elements of maximum weight.*

# A more interesting problem

- 1 Given  $n$  items  $N = \{e_1, e_2, \dots, e_n\}$ . Each item  $e_i$  has a non-negative weight  $w_i$ .
- 2 Items partitioned into  $h$  sets  $N_1, N_2, \dots, N_h$ . Think of each item having one of  $h$  colors.
- 3 Given integers  $k_1, k_2, \dots, k_h$  and another integer  $k$
- 4 Goal: pick  $k$  elements such that no more than  $k_i$  from  $N_i$  to **maximize** total weight of items picked.

	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$
<i>weight</i>	9	5	4	7	5	2	1

$$N_1 = \{e_1, e_2, e_3\}, N_2 = \{e_4, e_5\}, N_3 = \{e_6, e_7\}$$

$$k = 4, k_1 = 2, k_2 = 1, k_3 = 2$$

# Greedy Template

```
N is the set of all elements X  $\leftarrow \emptyset$   
(* X will store all the elements that will be picked *)  
while N is not empty do  
    N' = { $e_i \in N \mid X \cup \{e_i\}$  is feasible}  
    if N' =  $\emptyset$  then break  
    choose  $e_j \in N'$  of maximum weight  
    add  $e_j$  to X  
    remove  $e_j$  from N  
return the set X
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# Greedy Template

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N is the set of all elements  $X \leftarrow \emptyset$ 
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while N is not empty do
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```

## Theorem

*Greedy is optimal for the problem on previous slide.*

Proof: exercise after class.

Special case of general phenomenon of Greedy working for maximum weight independent set in a **matroid**. Beyond scope of course.

# Part V

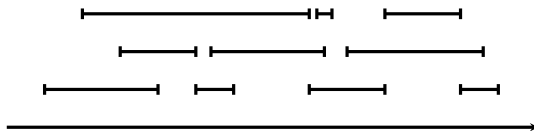
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## Problem (Interval Scheduling)

**Input:** *A set of jobs with start and finish times to be scheduled on a resource (example: classes and class rooms).*

**Goal:** *Schedule as many jobs as possible*



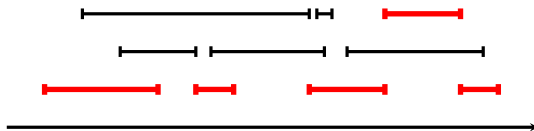
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**Input:** *A set of jobs with start and finish times to be scheduled on a resource (example: classes and class rooms).*

**Goal:** *Schedule as many jobs as possible*

- ① *Two jobs with overlapping intervals cannot both be scheduled!*



# Greedy Template

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R is the set of all requests
X ← ∅ (* X will store all the jobs that will be scheduled *)
while R is not empty do
    choose i ∈ R
    add i to X
    remove from R all requests that overlap with i
return the set X
```



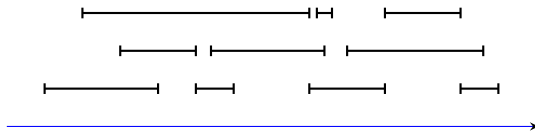
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    remove from  $R$  all requests that overlap with  $i$   
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**Main task:** Decide the order in which to process requests in  $R$

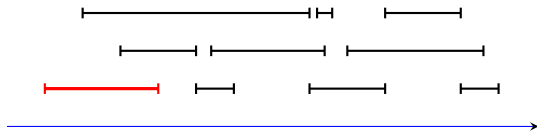
# Earliest Start Time

Process jobs in the order of their starting times, beginning with those that start earliest.



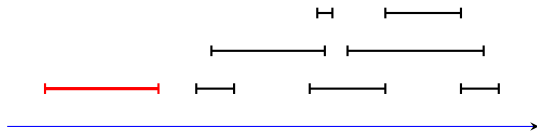
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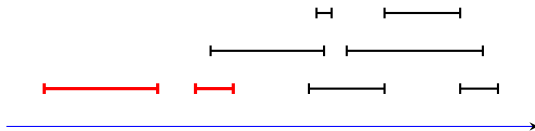
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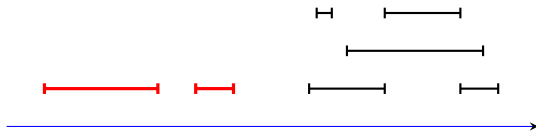
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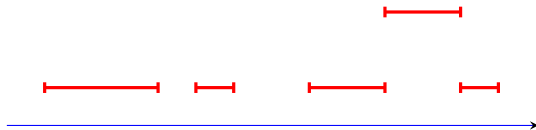
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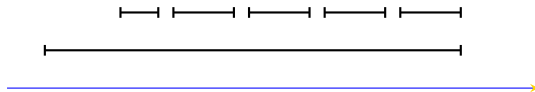


Figure: Counter example for earliest start time



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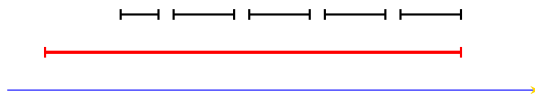


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Process jobs in the order of processing time, starting with jobs that require the shortest processing.



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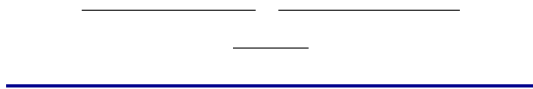


Figure: Counter example for smallest processing time



# Smallest Processing Time

Process jobs in the order of processing time, starting with jobs that require the shortest processing.

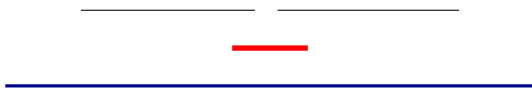


Figure: Counter example for smallest processing time

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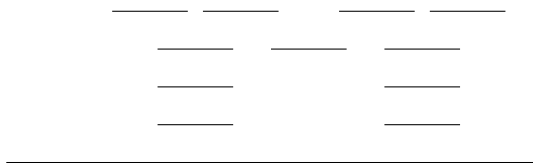


Figure: Counter example for fewest conflicts



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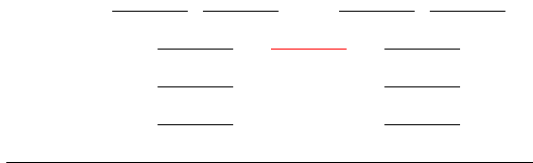


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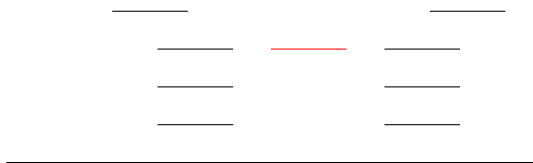
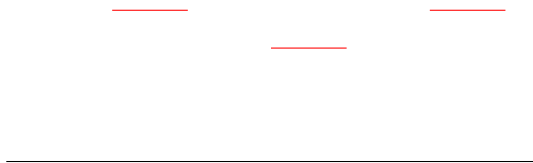


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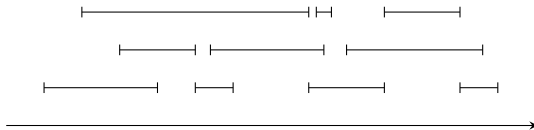
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**Figure:** Counter example for fewest conflicts

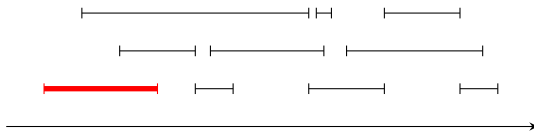
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Process jobs in the order of their finishing times, beginning with those that finish earliest.



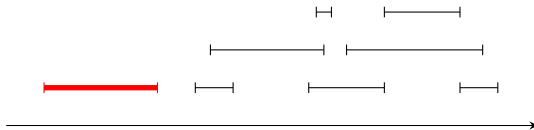
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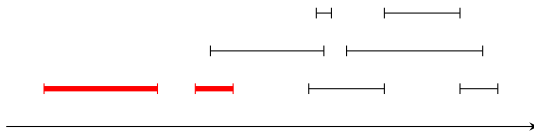
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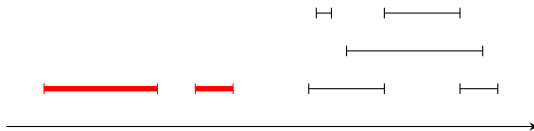
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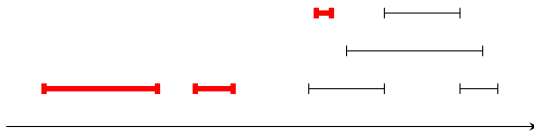
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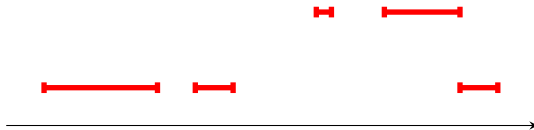
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# Optimal Greedy Algorithm

```
R is the set of all requests  
X  $\leftarrow \emptyset$  (* X stores the jobs that will be scheduled *)  
while R is not empty  
    choose  $i \in R$  such that finishing time of  $i$  is smallest  
    add  $i$  to X  
    remove from R all requests that overlap with  $i$   
return X
```

## Theorem

*The greedy algorithm that picks jobs in the order of their finishing times is optimal.*

# Proving Optimality

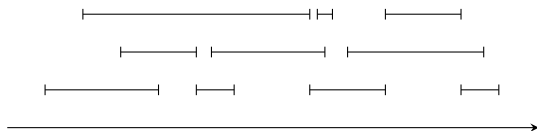
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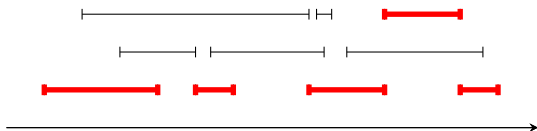
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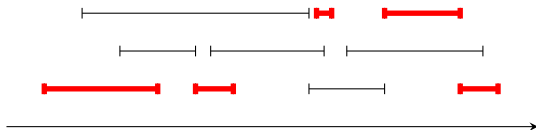
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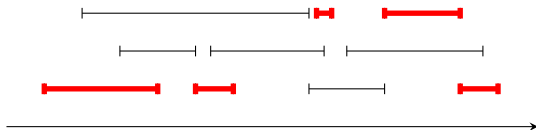
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Instead we will show that  $|O| = |X|$

# Proof of Optimality: Key Lemma

## Lemma

Let  $i_1$  be first interval picked by Greedy. There exists an optimum solution that contains  $i_1$ .

## Proof.

Let  $O$  be an *arbitrary* optimum solution. If  $i_1 \in O$  we are done.

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**Claim:** If  $i_1 \notin O$  then there is exactly one interval  $j_1 \in O$  that conflicts with  $i_1$ . (proof later)

- 1 Form a new set  $O'$  by removing  $j_1$  from  $O$  and adding  $i_1$ , that is  $O' = (O - \{j_1\}) \cup \{i_1\}$ .
- 2 From claim,  $O'$  is a *feasible* solution (no conflicts).
- 3 Since  $|O'| = |O|$ ,  $O'$  is also an optimum solution and it contains  $i_1$ . □

# Proof of Claim

## Claim

*If  $i_1 \notin O$ , there is exactly one interval  $j_1 \in O$  that conflicts with  $i_1$ .*

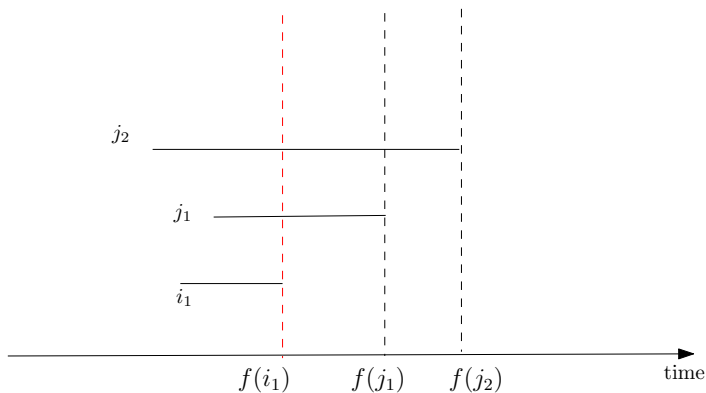
## Proof.

- 1 If no  $j \in O$  conflicts with  $i_1$  then  $O$  is not optimal!
- 2 Suppose  $j_1, j_2 \in O$  such that  $j_1 \neq j_2$  and both  $j_1$  and  $j_2$  conflict with  $i_1$ .
- 3 Since  $i_1$  has earliest finish time,  $j_1$  and  $i_1$  overlap at  $f(i_1)$ .
- 4 For same reason  $j_2$  also overlaps with  $i_1$  at  $f(i_1)$ .
- 5 Implies that  $j_1, j_2$  overlap at  $f(i_1)$  but intervals in  $O$  cannot overlap.

See figure in next slide.



# Figure for proof of Claim



**Figure:** Since  $i_1$  has the earliest finish time, any interval that conflicts with it does so at  $f(i_1)$ . This implies  $j_1$  and  $j_2$  conflict.

# Proof of Optimality of Earliest Finish Time First

## Proof by Induction on number of intervals.

**Base Case:**  $n = 1$ . Trivial since Greedy picks one interval.

**Induction Step:** Assume theorem holds for  $i < n$ .

Let  $I$  be an instance with  $n$  intervals

$I'$ :  $I$  with  $i_1$  and all intervals that overlap with  $i_1$  removed

$G(I), G(I')$ : Solution produced by Greedy on  $I$  and  $I'$

From Lemma, there is an optimum solution  $O$  to  $I$  and  $i_1 \in O$ .

Let  $O' = O - \{i_1\}$ .  $O'$  is a solution to  $I'$ .

$$\begin{aligned} |G(I)| &= 1 + |G(I')| \quad (\text{from Greedy description}) \\ &\geq 1 + |O'| \quad (\text{By induction, } G(I') \text{ is optimum for } I') \\ &= |O| \end{aligned}$$



# Implementation and Running Time

```
Initially  $R$  is the set of all requests
 $X \leftarrow \emptyset$  (*  $X$  stores the jobs that will be scheduled *)
while  $R$  is not empty
    choose  $i \in R$  such that finishing time of  $i$  is least
    if  $i$  does not overlap with requests in  $X$ 
        add  $i$  to  $X$ 
    remove  $i$  from  $R$ 
return the set  $X$ 
```

- Presort all requests based on finishing time.  $O(n \log n)$  time
- Now choosing least finishing time is  $O(1)$
- Keep track of the finishing time of the last request added to  $A$ . Then check if starting time of  $i$  later than that
- Thus, checking non-overlapping is  $O(1)$
- Total time  $O(n \log n + n) = O(n \log n)$



# Comments

- ① Interesting Exercise: smallest interval first picks at least half the optimum number of intervals.
- ② All requests need not be known at the beginning. Such *online* algorithms are a subject of research

# Weighted Interval Scheduling

Suppose we are given  $n$  jobs. Each job  $i$  has a start time  $s_i$ , a finish time  $f_i$ , and a weight  $w_i$ . We would like to find a set  $S$  of compatible jobs whose total weight is maximized. Which of the following greedy algorithms finds the optimum schedule?

- (A) Earliest start time first.
- (B) Earliest finish time first.
- (C) Highest weight first.
- (D) None of the above.
- (E) IDK.

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Weighted problem can be solved via dynamic programming. See notes.

# Greedy Analysis: Overview

- ① **Greedy's first step leads to an optimum solution.** Show that there is an optimum solution leading from the first step of Greedy and then use induction. Example, Interval Scheduling.
- ② **Greedy algorithm stays ahead.** Show that after each step the solution of the greedy algorithm is at least as good as the solution of any other algorithm. Example, Interval scheduling.
- ③ **Structural property of solution.** Observe some structural bound of every solution to the problem, and show that greedy algorithm achieves this bound. Example, Interval Partitioning (see Kleinberg-Tardos book).
- ④ **Exchange argument.** Gradually transform any optimal solution to the one produced by the greedy algorithm, without hurting its optimality. Example, Minimizing lateness.

# Takeaway Points

- ① Greedy algorithms come naturally but often are incorrect. A proof of correctness is an absolute necessity.
- ② *Exchange* arguments are often the key proof ingredient. Focus on why the first step of the algorithm is correct: need to show that there is an optimum/correct solution with the first step of the algorithm.
- ③ Thinking about correctness is also a good way to figure out which of the many greedy strategies is likely to work.