

Give context-free grammars for each of the following languages.

1 $\{0^{2n}1^n \mid n \geq 0\}$

■ **Solution:** $S \rightarrow \varepsilon \mid 00S1$.

2 $\{0^m1^n \mid m \neq 2n\}$

(Hint: If $m \neq 2n$, then either $m < 2n$ or $m > 2n$.)

Solution:

To simplify notation, let $\Delta(w) = \#(0, w) - 2\#(1, w)$. Our solution follows the following logic. Let w be an arbitrary string in this language.

- Because $\Delta(w) \neq 0$, then either $\Delta(w) > 0$ or $\Delta(w) < 0$.
- If $\Delta(w) > 0$, then $w = 0^i z$ for some integer $i > 0$ and some suffix z with $\Delta(z) = 0$.
- If $\Delta(w) < 0$, then $w = x1^j$ for some integer $j > 0$ and some prefix x with either $\Delta(x) = 0$ or $\Delta(x) = 1$.
- Substrings with $\Delta = 0$ is generated by the previous grammar; we need only a small tweak to generate substrings with $\Delta = 1$.

Here is one way to encode this case analysis as a CFG. The nonterminals M and L generate all strings where the number of 0s is More or Less than twice the number of 1s, respectively. The last nonterminal generates strings with $\Delta = 0$ or $\Delta = 1$.

$$\begin{array}{ll} S \rightarrow M \mid L & \{0^m1^n \mid m \neq 2n\} \\ M \rightarrow 0M \mid 0E & \{0^m1^n \mid m > 2n\} \\ L \rightarrow L1 \mid E1 & \{0^m1^n \mid m < 2n\} \\ E \rightarrow \varepsilon \mid 0 \mid 00E1 & \{0^m1^n \mid m = 2n \text{ or } 2n + 1\} \end{array}$$

Here is a different correct solution using the same logic. We either identify a non-empty prefix of 0s or a non-empty prefix of 1s, so that the rest of the string is as “balanced” as possible. We also generate strings with $\Delta = 1$ using a separate non-terminal.

$$\begin{array}{ll} S \rightarrow AE \mid EB \mid FB & \{0^m1^n \mid m \neq 2n\} \\ A \rightarrow 0 \mid 0A & 0^+ = \{0^i \mid i \geq 1\} \\ B \rightarrow 1 \mid 1B & 1^+ = \{1^j \mid j \geq 1\} \\ E \rightarrow \varepsilon \mid 00E1 & \{0^m1^n \mid m = 2n\} \\ F \rightarrow 0E & \{0^m1^n \mid m = 2n + 1\} \end{array}$$

Alternatively, we can separately generate all strings of the form $0^{\text{odd}}1^*$, so that we don't have to worry about the case $\Delta = 1$ separately.

$$\begin{array}{ll} S \rightarrow D \mid M \mid L & \{0^m1^n \mid m \neq 2n\} \\ D \rightarrow 0 \mid 00D \mid D1 & \{0^m1^n \mid m \text{ is odd}\} \\ M \rightarrow 0M \mid 0E & \{0^m1^n \mid m > 2n\} \\ L \rightarrow L1 \mid E1 & \{0^m1^n \mid m < 2n \text{ and } m \text{ is even}\} \\ E \rightarrow \varepsilon \mid 00E1 & \{0^m1^n \mid m = 2n\} \end{array}$$

Solution:

Intuitively, we can parse any string $w \in L$ as follows. First, remove the first $2k$ 0s and the last k 1s, for the largest possible value of k . The remaining string cannot be empty, and it must consist entirely of 0s, entirely of 1s, or a single 0 followed by 1s.

$$\begin{array}{ll} S \rightarrow 00S1 \mid A \mid B \mid C & \{0^m 1^n \mid m \neq 2n\} \\ A \rightarrow 0 \mid 0A & 0^+ \\ B \rightarrow 1 \mid 1B & 1^+ \\ C \rightarrow 0 \mid 0B & 01^+ \end{array}$$

Lets elaborate on the above, since k is maximal, $w = 0^{2k}w'1^k$. If w' starts with 00, and ends with a 1, then we can increase k by one. As such, w' is either in 0^+ or 1^+ . If w' contains both 0s and 1s, then it can contain only a single 0, followed potentially by 1^+ . We conclude that $w' \in 0^+ + 1^+ + 01^+$.

3 $\{0,1\}^* \setminus \{0^{2n}1^n \mid n \geq 0\}$

Solution:

This language is the union of the previous language and the complement of 0^*1^* , which is $(0+1)^*10(0+1)^*$.

$$\begin{array}{ll} S \rightarrow T \mid X & \{0,1\}^* \setminus \{0^{2n}1^n \mid n \geq 0\} \\ T \rightarrow 00T1 \mid A \mid B \mid C & \{0^m 1^n \mid m \neq 2n\} \\ A \rightarrow 0 \mid 0A & 0^+ \\ B \rightarrow 1 \mid 1B & 1^+ \\ C \rightarrow 0 \mid 0B & 01^+ \\ X \rightarrow Z10Z & (0+1)^*10(0+1)^* \\ Z \rightarrow \varepsilon \mid 0Z \mid 1Z & (0+1)^* \end{array}$$

Work on these later:

4 $\{w \in \{0,1\}^* \mid \#(0,w) = 2 \cdot \#(1,w)\}$ – Binary strings where the number of 0s is exactly twice the number of 1s.

Solution:

$$S \rightarrow \varepsilon \mid SS \mid 00S1 \mid 0S1S0 \mid 1S00.$$

Here is a sketch of a correctness proof; a more detailed proof appears in the homework.

For any string w , let $\Delta(w) = \#(0,w) - 2 \cdot \#(1,w)$. Suppose w is a binary string such that $\Delta(w) = 0$. Suppose w is nonempty and has no non-empty proper prefix x such that $\Delta(x) = 0$. There are three possibilities to consider:

- Suppose $\Delta(x) > 0$ for every proper prefix x of w . In this case, w must start with 00 and end with 1. Thus, $w = 00x1$ for some string $x \in L$.

- Suppose $\Delta(x) < 0$ for every proper prefix x of w . In this case, w must start with **1** and end with **00**. Let x be the shortest non-empty prefix with $\Delta(x) = 1$. Thus, $w = 1X00$ for some string $x \in L$.
- Finally, suppose $\Delta(x) > 0$ for some prefix x and $\Delta(x') < 0$ for some longer proper prefix x' . Let x' be the shortest non-empty proper prefix of w with $\Delta < 0$. Then $x' = 0y1$ for some substring y with $\Delta(y) = 0$, and thus $w = 0y1z0$ for some strings $y, z \in L$.

5 $\{0, 1\}^* \setminus \{ww \mid w \in \{0, 1\}^*\}$.

Solution:

All strings of odd length are in L .

Let w be any even-length string in L , and let $m = |w|/2$. For some index $i \leq m$, we have $w_i \neq w_{m+i}$. Thus, w can be written as either $x1y0z$ or $x0y1z$ for some substrings x, y, z such that $|x| = i - 1$, $|y| = m - 1$, and $|z| = m - i$. We can further decompose y into a prefix of length $i - 1$ and a suffix of length $m - i$. So we can write any even-length string $w \in L$ as either $x1x'z'0z$ or $x0x'z'1z$, for some strings x, x', z, z' with $|x| = |x'| = i - 1$ and $|z| = |z'| = m - i$. Said more simply, we can divide w into two odd-length strings, one with a **0** at its center, and the other with a **1** at its center.

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| $S \rightarrow AB \mid BA \mid A \mid B$ | strings not of the form ww |
| $A \rightarrow 0 \mid \Sigma A \Sigma$ | odd-length strings with 0 at center |
| $B \rightarrow 1 \mid \Sigma B \Sigma$ | odd-length strings with 1 at center |
| $\Sigma \rightarrow 0 \mid 1$ | single character |