Proving Non-regularity

Lecture 7
Feb 11, 2020

Regular Languages, DFAs, NFAs

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Languages accepted by DFAs, NFAs, and regular expressions are the same.

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- Hence number of regular languages is *countably infinite*
- Number of languages is *uncountably infinite*
- Hence there must be a non-regular language!

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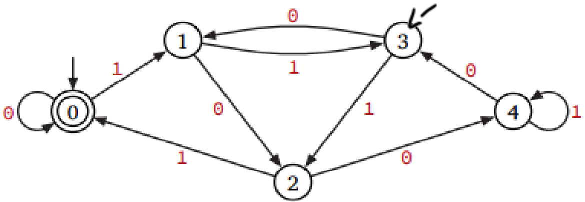
How do we formalize intuition and come up with a formal proof?

Intuition: How DFA Works



For $x = 11$ and $y = 1000$

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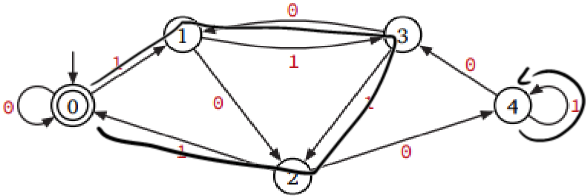

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What is the behavior of M on these strings? Let $q_i = \delta^*(s, 0^i)$.

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M should accept $0^i 1^i$ but then it will also accept $0^j 1^i$ where $i \neq j$.
This contradicts the fact that M accepts L. Thus, there is no DFA for L.

Generalizing the argument

Definition

For a language L over Σ and two strings $x, y \in \Sigma^*$ we say that x and y are distinguishable with respect to L if there is a string $w \in \Sigma^*$ such that exactly one of xw, yw is in L. In other words either $xw \in L, yw \notin L$ or $xw \notin L, yw \in L$.

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Example: 000 and 0000 are indistinguishable with respect to the language $L = \{w \mid w \text{ has } 00 \text{ as a substring}\}$

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Lemma

Suppose $L = L(M)$ *for some DFA* $M = (Q, \Sigma, \delta, s, A)$ *and suppose* x, y *are distinguishable with respect to* L. *Then* $\delta^*(s, x) \neq \delta^*(s, y)$.

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Proof.

Since x, y are distinguishable let w be the distinguishing suffix.

If $\delta^*(s, x) = \delta^*(s, y)$ then M will either accept both the strings xw, yw, or reject both. But exactly one of them is in L, a contradiction. □

Fooling Sets

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Example 2: Multiple of 5.

$x = 0^{10},\ y = 0^{100} \in F$

$w = 1^{10}$

$0^{10}1^{10} \in L$

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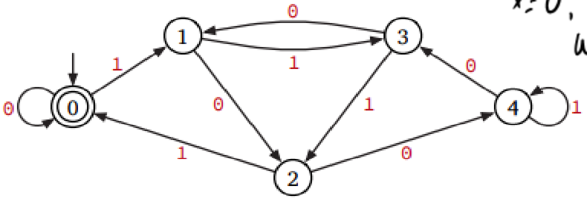

$F = \{0, 1, 11, 10, 100\}$ → 0, 1, 3, 2, 4

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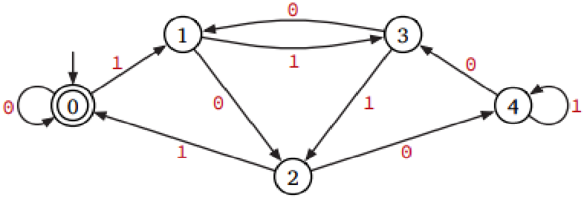

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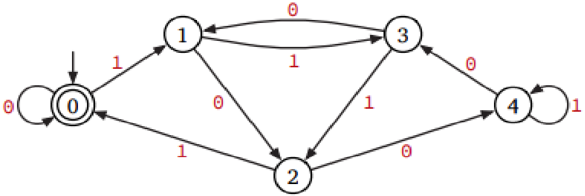

$F = \{0, 1, 10, 11, 100\}$. Can we add more to this set?

Fooling Set Size vs Size of DFA

Theorem

Suppose F is a fooling set for L. If F is finite then there is no DFA M that accepts L with less than $|F|$ states.

Proof.

Suppose there is a DFA $M = (Q, \Sigma, \delta, s, A)$ that accepts L. Let $|Q| = n$.

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Implies that there is w such that exaclty one of xw, yw is in L. However, M's behaviour on xw and yw is exacly the same and hence M will accept both xw, yw or reject both. A contradiction. □

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Proof.

Suppose for contradiction that $L = L(M)$ for some DFA M with n states.
Any subset F' of F is a fooling set. (Why?) Pick $F' \subseteq F$ arbitrarily such that $|F'| > n$. By preceding theorem, we obtain a contradiction. □

Examples

- L_1 $\{0^k1^k \mid k \geq 0\}$

  $F_1 = \{0^*\}$ $(0^i, 0^j) \in F \times F$ $w = 1^i$

- L_2 {bitstrings with equal number of 0s and 1s}

  1010 110100 $F_2 = \{(001)^*\}$ $F_2 = \{0^*\}$

- $\{0^k1^\ell \mid k \neq \ell\}$

  $F_3 = 0^*$

  $3 \leq i < j$

- $\{0^{k^2} \mid k \geq 0\}$

  $F_4 = \{0^k \mid k \geq 3\}$ $(0^i, 0^j)$ $w = 0^{j^2 - j}$

  $0^i 0^{j^2 - j}$ $\nexists k, j^2 - j + i = k^2$

How to pick a fooling set

How do we pick a fooling set F?

- If x, y are in F and $x \neq y$ they should be distinguishable! Of course.
- All strings in F except maybe one should be prefixes of strings in the language L.
  For example if $L = \{0^k 1^k \mid k \geq 0\}$ do not pick 1 and 10 (say). Why?

Part I

Non-regularity via closure properties

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$L = \{$bitstrings with equal number of 0s and 1s$\}$

$L' = \{0^k 1^k \mid k \geq 0\}$

Suppose we know that L' is non-regular. Can we show that L is non-regular without using the fooling set argument from scratch?

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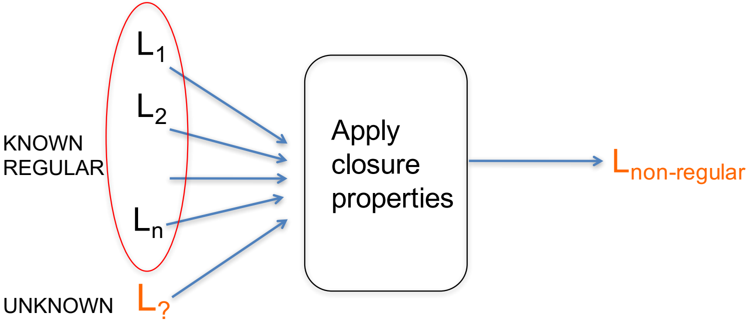
Claim: The above and the fact that L' is non-regular implies L is non-regular. Why?

Suppose L is regular. Then since $L(0^*1^*)$ is regular, and regular languages are closed under intersection, L' also would be regular. But we know L' is not regular, a contradiction.

Non-regularity via closure properties

General recipe:



Proving non-regularity: Summary

- DFAs have fixed memory. Any language that requires memory that grows with input size is not regular. Not always easy to tell!
- Method of distinguishing suffixes. To prove that L is non-regular find an infinite fooling set.
- Closure properties. Use existing non-regular languages and regular languages to prove that some new language is non-regular.
- Pumping lemma. We did not cover it but it is sometimes an easier proof technique to apply, but not as general as the fooling set technique.

Part II

Myhill-Nerode Theorem

Intuition: DFA size vs Fooling set

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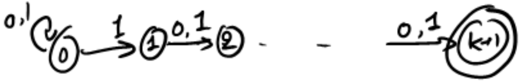

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Why?

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Why?

- Suppose $a_1 a_2 \ldots a_k$ and $b_1 b_2 \ldots b_k$ are two distinct bitstrings of length k
- Let i be first index where $a_i \neq b_i$
- $y = 0^{i-1}$ is a distinguishing suffix for the two strings

Indistinguishability

Recall:

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Given language L over Σ define a relation $\equiv_L$ over strings in Σ^* as follows: $x \equiv_L y$ iff x and y are indistinguishable with respect to L.

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Claim

Let x, y *be two distinct strings. If* x, y *belong to the same equivalence class of* $\equiv_L$ *then* x, y *are indistinguishable. Otherwise they are distinguishable.*

Corollary

If $\equiv_L$ *is finite with* n *equivalence classes then there is a fooling set* F *of size* n *for* L. *If* $\equiv_L$ *is infinite then there is an infinite fooling set for* L.

Myhill-Nerode Theorem

Theorem (Myhill-Nerode)

L is is regular if and only if $\equiv_L$ has a finite number of equivalence classes. If $\equiv_L$ is finite with n equivalence classes then there is a DFA M accepting L with exactly n states and this is the minimum possible.

Corollary

A language L is non-regular if and only if there is an infinite fooling set F for L.

Algorithmic implication: For every DFA M one can find in polynomial time a DFA M' such that $L(M) = L(M')$ and M' has the fewest possible states among all such DFAs.