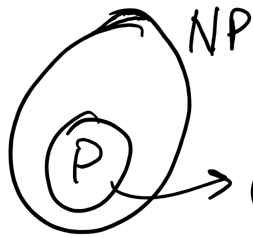
Lec 27: Proving NP-completeness

Saturday, April 25, 2020 7:18 PM

Last Lec:



→ poly-time computable

→ NP-Complete (NPC) (Hardest problems in NP)

→ SAT is NP-complete

→ 3SAT is NP-complete

Method for Proving NP-completeness: Decision Prob. L

1. $L \in NP$ (in NP)
2. $L^* \le_p L$, where L^* is NP-complete (NP-hardness for L).

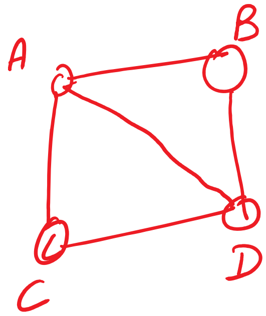
☆ Independent Set (IS):

I/P: Undirected graph $G=(V,E)$, int k

O/P: YES iff G has IS of size $\ge k$

iff $\exists S \subseteq V$, $|S| \ge k$ & $u, v \in S \Rightarrow (u,v) \notin E$.

e.g.



$\{B, C\}$ is an IS.

... complete.

C D

Thm: Independent Set is NP-Complete.

Pf: (1) IS $\in$ NP

certificate: $S \subseteq V$

certifier: (1) $|S| \geq k$

(2) $(u,v) \in E \Rightarrow u \notin S$ or $v \notin S$

Poly-time.

(2) IS is NP-hard.

Claim: $\boxed{3SAT \leq_p IS.}$

Pf: Given an i/p to 3SAT: Boolean formula on $x_1, \ldots x_n$

$$\boxed{F = \bigwedge_{i=1}^{m} \underbrace{(\alpha_{i1} \vee \alpha_{i2} \vee \alpha_{i3})}_{\text{clause } C_i}}$$

each α_{ij} is either a variable or it's negation.

literal $\quad \alpha_{ij} = x_d$ or $\bar{x}_d$ for some var x_d.

O/P: YES if $\exists$ boolean assignment $x_i \in \{0,1\}$ $\forall i$

that makes F true.

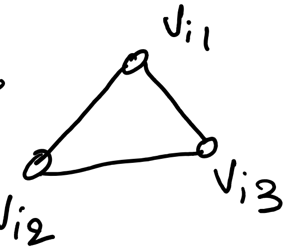
★ Given i/p of 3SAT: Boolean formula F, Construct an i/p for indep. set: undirected graph $G=(V,E)$, int $\underline{k}$.

as follows:

As follows:

For each clause $C_i = (\alpha_{i1} \vee \alpha_{i2} \vee \alpha_{i3})$

→ Create 3 vertices v_{i1}, v_{i2}, v_{i3}



→ create 3 edges

$(v_{i1}, v_{i2}), (v_{i2}, v_{i3})\ (v_{i3}, v_{i1})$

Intuition: if $\underline{\alpha_{ij}} = x_d$, $\underline{\alpha_{i'j'}} = \bar{x}_d$

Then both α_{ij} & $\alpha_{i'j'}$ can not be true

→ "Cross-edges"

If $\alpha_{ij} = \overline{\alpha}_{i'j'}$ then $(\underline{v_{ij}}, v_{i'j'}) \in E$.

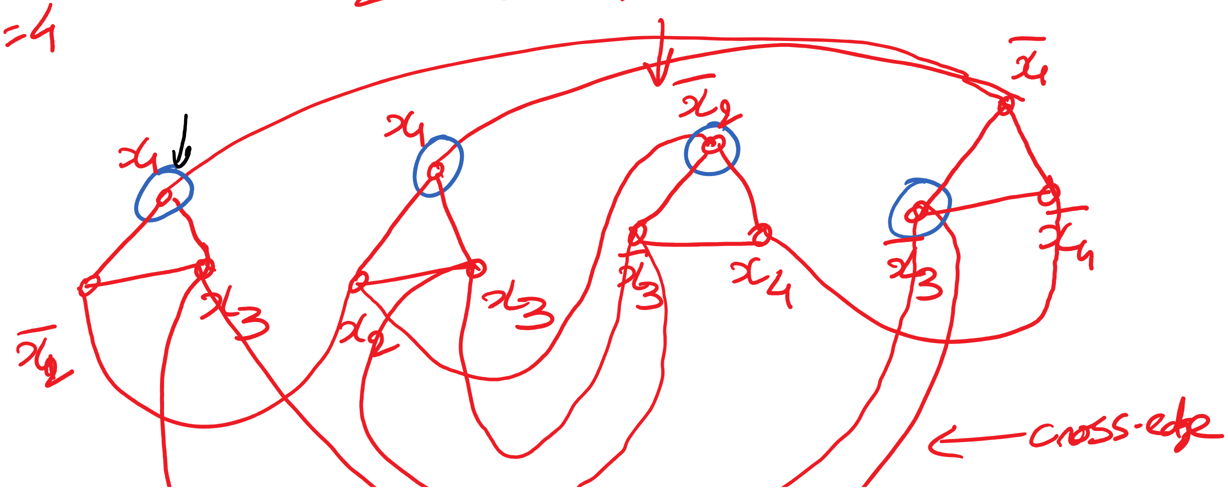
$\boxed{K = m}$

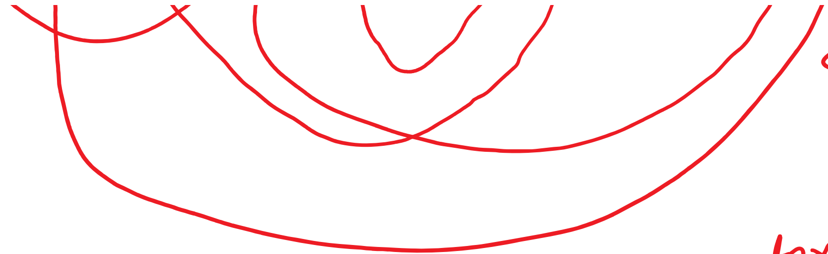
$\underline{(x_1 = 1,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0)}$

e.g. $F = (\underline{x_1} \vee \bar{x}_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$

$\wedge (\bar{x}_2 \vee \bar{x}_3 \vee x_4) \wedge (\underline{\bar{x}_1} \vee \bar{x}_3 \vee \bar{x}_4)$

$K = 4$



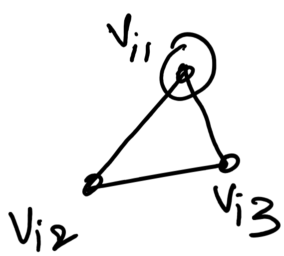


obs 1: Can pick at most 1 vertex per triangle ≡ a clause

obs 2: If pick vertex corresponding to x_d then cannot pick any vertex corr to $\bar{x}_d$.

Correctness : $\exists$ assignment A that makes F true $\Longleftrightarrow$ $\exists$ an indep. set $S \subseteq V$ of size $\geq k = m$ in G.

Pf : ($\Rightarrow$) Given a true assignment A of F. → construct S.



For each clause $(\alpha_{i1} \vee \alpha_{i2} \vee \alpha_{i3})$

at least one of $\alpha_{i1}, \alpha_{i2}, \alpha_{i3}$ is true

Pick exactly one true α_{ij}

& put corr. v_{ij} in S.

— —

then both α_{ij} & $\alpha_{i'j'}$ can not be true

$\Rightarrow$ both v_{ij} & $v_{i'j'} \notin S$

$\Rightarrow (v_{ij}, v_{i'j'}) \in E \Rightarrow$ either $v_{ij} \notin S$
or $v_{i'j'} \notin S$

$|S| = m \geq K$

($\Leftarrow$) Given an indep. set S of size $\geq m$
in graph G

construct an assignment A for F as

follows:

If $v_{ij} \in S$ then set α_{ij} to true ind.

$\boxed{\begin{array}{l} \text{If } \alpha_{ij} = x_d \text{ then } x_d = 1 \\ \alpha_{ij} = \overline{x_d} \text{ then } x_d = 0 \end{array}}$

$\rightarrow$ Is A consistent? YES:

If $\alpha_{ij} = \overline{\alpha_{i'j'}}$ then $(v_{ij}, v_{i'j'})$

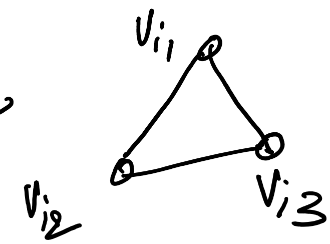
$\Rightarrow$ at most one of v_{ij} & $v_{i'j'}$ in S.

A will not set both α_{ij} & $\alpha_{i'j'}$ to true.

$\rightarrow$ Does A makes $\underline{F}$ true? YES:

For each clause $C_i = (\ell_{i1} \vee \ell_{i2} \vee \ell_{i3})$

at most one of v_{i1}, v_{i2}, v_{i3} in S



Since $|S| \geq m$, it must be $|S| = m$

Then exactly one $v_{ij} \in S$ for each clause i.

$\Rightarrow$ the corr. ℓ_{ij} is true in A for each clause C_i. ∎

Corr: IS $\leq_p$ Vertex Cover $\Rightarrow$ Vertex cover is NPC

$\underbrace{\text{Vertex Cover} \leq_p \text{Set Cover}}_{\text{NP-hardness}}$ $\Rightarrow$ set cover " "

(In NP: Exercise)

☆ Hamiltonian Cycle:

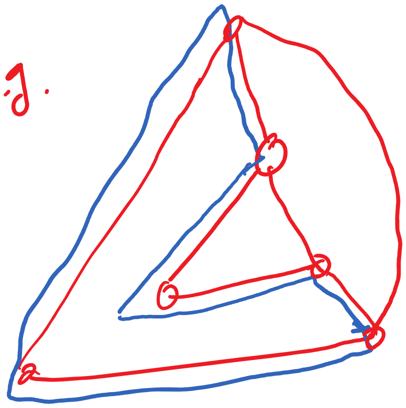
I/P: (un)directed graph $G = (V, E)$

O/P: YES iff $\exists$ a Hamiltonian cycle 'C' in G.

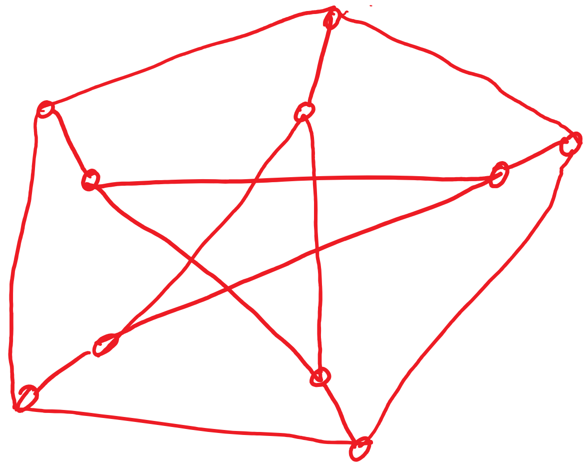
iff C visits each vertex "exactly" once.

once.

e.g.

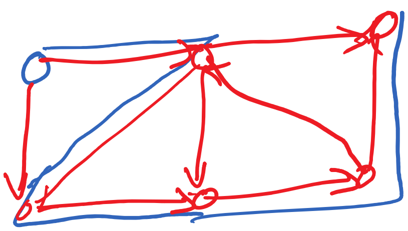


YES.



NO.

dir. graphs

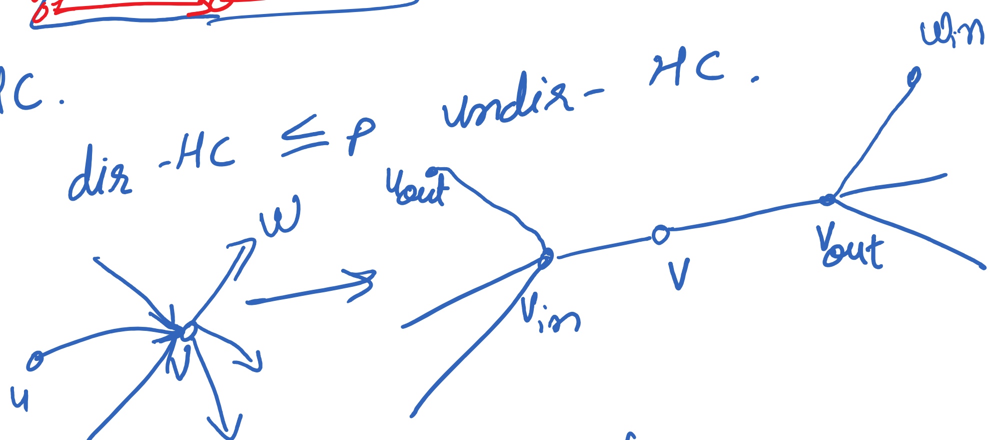


Variant: Ham. Path?

NO to HC.

$\text{dir-HC} \le_P \text{undir-HC}$.

Remark:



poly. time reduction.

If dir-HC is NP-Complete then undir-HC is " .

Thm (Karp '71): Hamiltonian cycle is NP-Complete

Pf: (1) HC $\in$ NP (exercise). } dir-HC is NPC

(2) ... NP-hard.

Pf: (1) HC $\in$ NP
(2) dir-HC is NP-hard. } is NPC

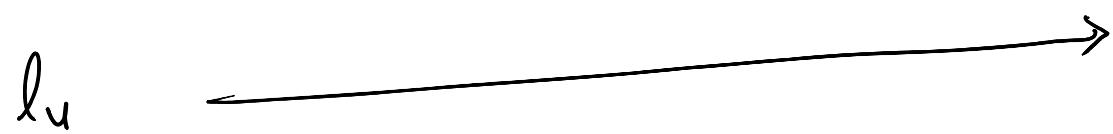
Claim: Vertex-cover $\leq_p$ dir-HC.

Pf: Given i/p to vertex-cover: undirected graph $G=(V,E)$, int k

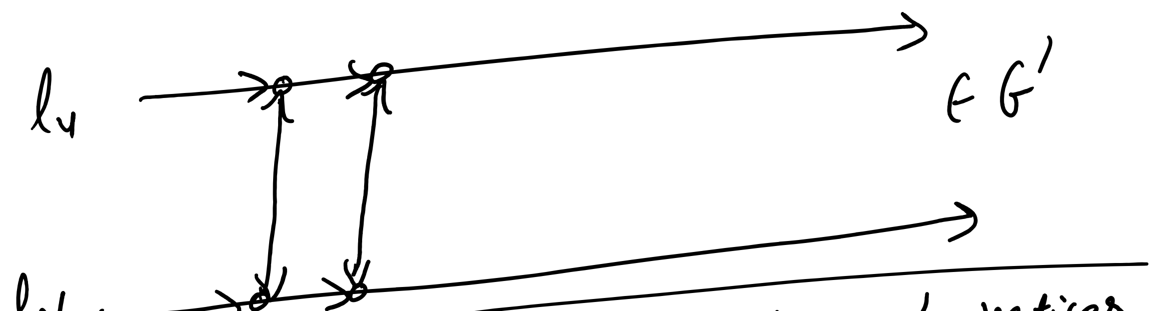
VC: $S \subseteq V$, $|S| \leq k$ & $(u,v) \in E \Rightarrow u \in S$ or $v \in S$ (Recall)

Construct i/p to dir-Ham. Cycle: directed graph G' as follows:

→ $u \in V$ "draw a line" l_u



→ each edge $(u,v) \in E$, construct "gadget".



Intuition: → covering these 4 vertices in cycle C of G' $\equiv$ covering edge (u,v)

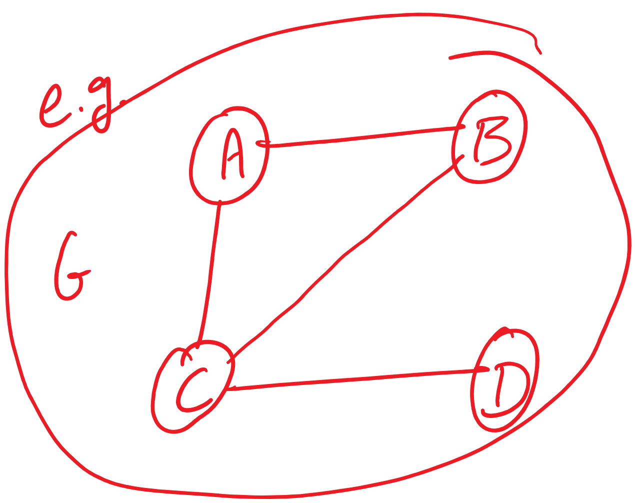
covering edge (u,v) in G.

$\to$ want $u \in S \iff$ traverse l_u

(To switch betⁿ lines)

- introduce vertices $z_1, \ldots z_k \in G'$
- introduce edge from end of each line to z_i & from z_i to start of each line.

e.g.

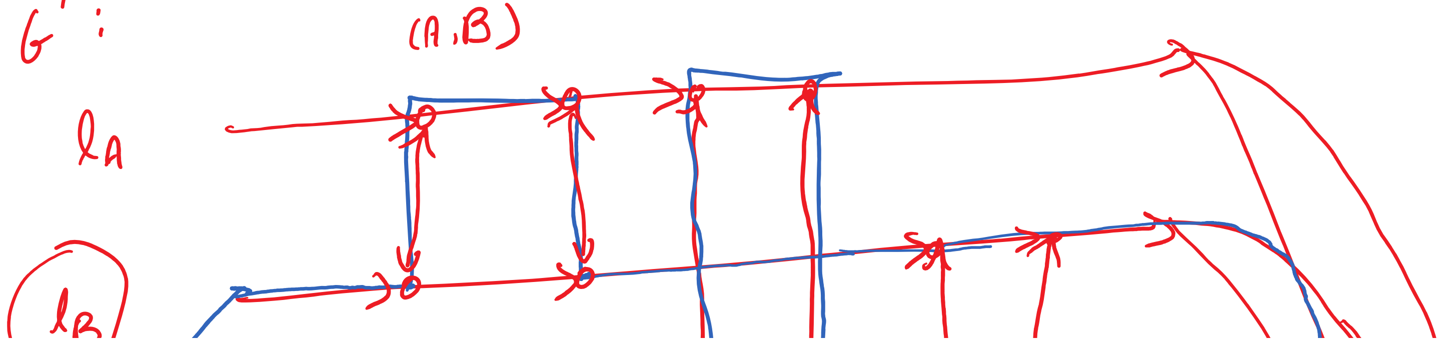


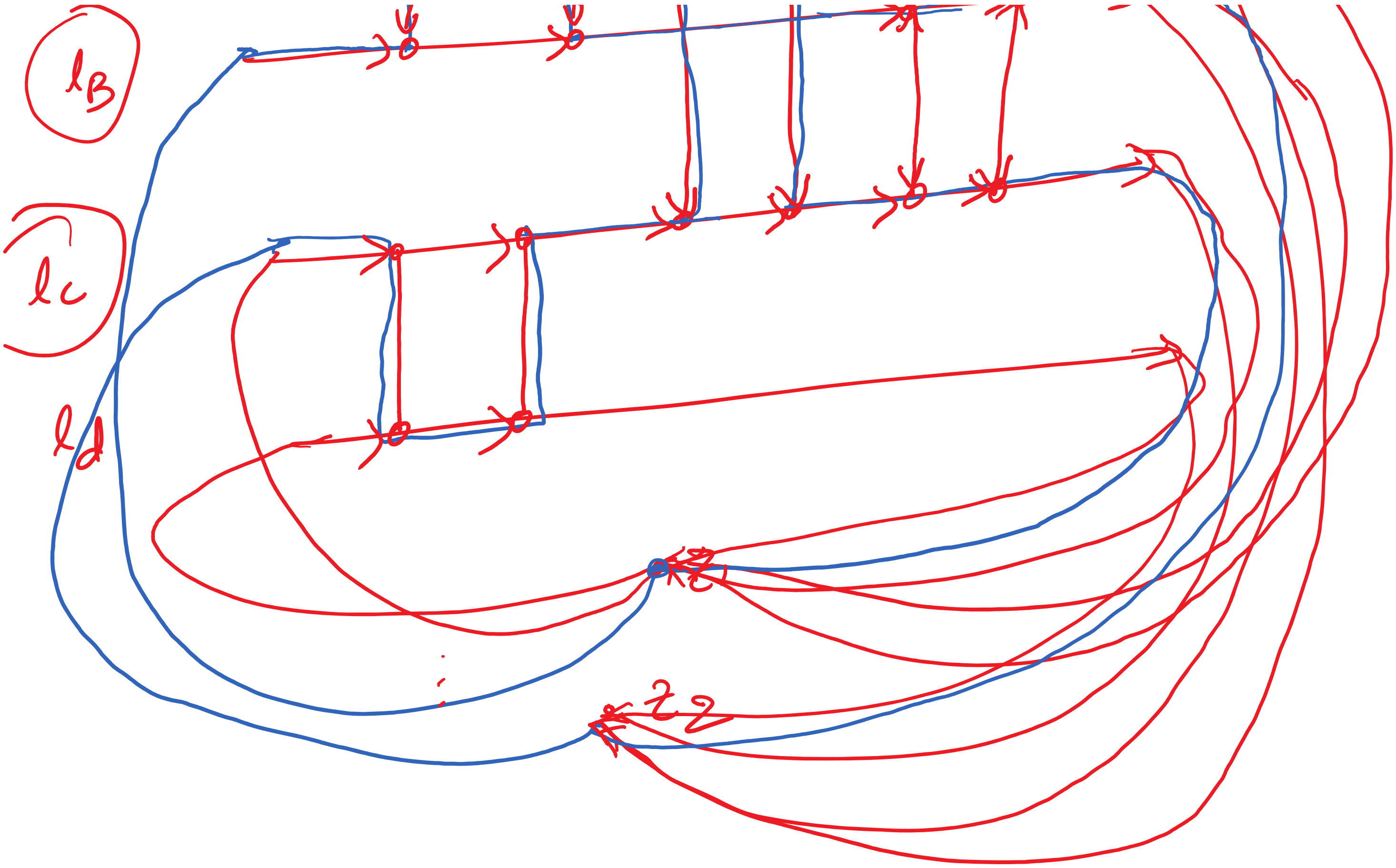
$K = 2$

$\{B, C\}$ is a VC.

$\downarrow$

G':





$\underline{a} \le_p \underline{b}$

i/p of a $\longrightarrow$ construct i/p of b

$I_a \longrightarrow I_b$

I_a is YES instance of Π $\Longleftrightarrow$ I_b YES instance of b

$\exists$ a sol^n for I_a $\Longleftrightarrow$ $\exists$ a sol^n for I_b