Breadth First Search, Dijkstra's Algorithm for Shortest Paths

Lecture 19

March 31, 2020

Part I

Breadth First Search

Breadth First Search (BFS)

Overview

(A) **BFS** is obtained from **BasicSearch** by processing ***ToExplore*** list as a **queue**.

(B) It processes the vertices in the order of their shortest distance from the vertex s (the start vertex).

As such...

1. **DFS** good for exploring graph structure
2. **BFS** good for exploring *distances*

Queue Data Structure

Queues

Queue is a **first-in first-out (FIFO)** list, i.e., elements are picked in the order in which they were inserted. Operations supported:

1. **enqueue**: Adds an element to the end of the list
2. **dequeue**: Removes an element from the front of the list

BFS Algorithm

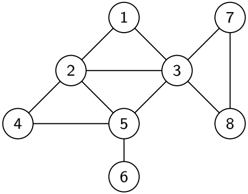
Given (undirected or directed) graph $G = (V, E)$ and node $s \in V$

BFS(s)
```
Mark all vertices as unvisited
Initialize search tree T to be empty
set Q to be the empty queue
Mark vertex s as visited.  enq(s)
while Q is nonempty do
    u = deq(Q)
    for each vertex v ∈ Adj(u)
        if v is not visited then
            add edge (u, v) to T
            Mark v as visited and enq(v)
```

Proposition

BFS(s) *runs in* $O(n + m)$ *time.*

BFS: An Example in Undirected Graphs



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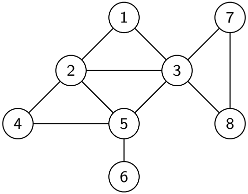

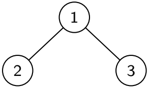

1. [1]
2. [2,3]

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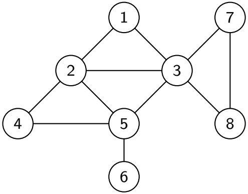

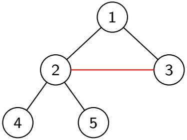

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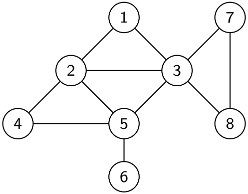

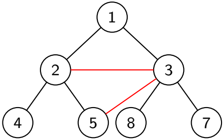

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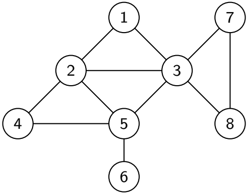

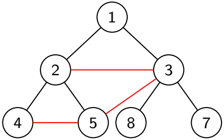

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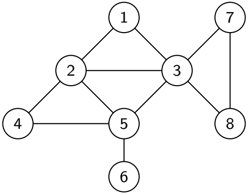

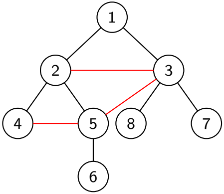

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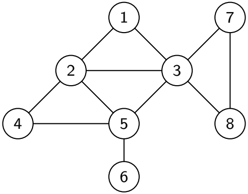

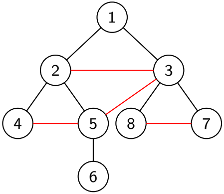

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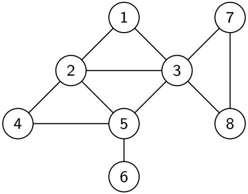

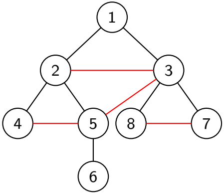

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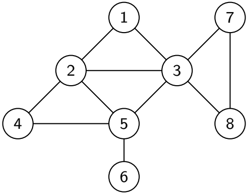

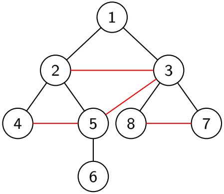

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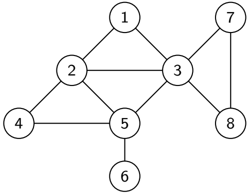

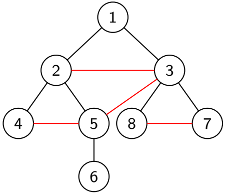

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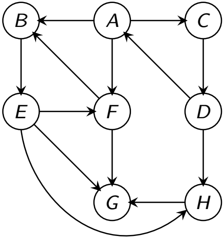
BFS tree is the set of black edges.

BFS: An Example in Directed Graphs



BFS with Distance

BFS(s)
    Mark all vertices as unvisited; for each v set $\mathrm{dist}(v) = \infty$
    Initialize search tree T to be empty
    set Q to be the empty queue
    Mark vertex s as visited and set $\mathrm{dist}(s) = 0$
    enq(s)
    while Q is nonempty **do**
        $u =$ **deq**(Q)
        for each vertex $v \in \mathrm{Adj}(u)$ **do**
            if v is not visited **do**
                add edge (u, v) to T
                Mark v as visited, **enq**(v)
                and set $\mathrm{dist}(v) = \mathrm{dist}(u) + 1$

Properties of BFS: Undirected Graphs

Theorem

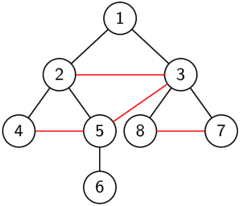
The following properties hold upon termination of **BFS**(s)

(A) *The search tree contains exactly the set of vertices in the connected component of* s.

(B) *If* $\mathrm{dist}(u) < \mathrm{dist}(v)$ *then* u *is visited before* v.

(C) *For every vertex* u, $\mathrm{dist}(u)$ *is the length of a shortest path (in terms of number of edges) from* s *to* u.

(D) *If* u, v *are in connected component of* s *and* $e = \{u, v\}$ *is an edge of* G, *then* $|\mathrm{dist}(u) - \mathrm{dist}(v)| \leq 1$.



Properties of BFS: Directed Graphs

Theorem

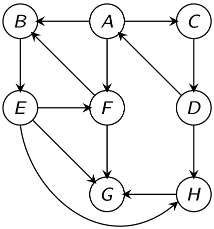
The following properties hold upon termination of **BFS**(s):

(A) *The search tree contains exactly the set of vertices reachable from* s

(B) *If* $\mathrm{dist}(u) < \mathrm{dist}(v)$ *then* u *is visited before* v

(C) *For every vertex* u, $\mathrm{dist}(u)$ *is indeed the length of shortest path from* s *to* u

(D) *If* u *is reachable from* s *and* $e = (u, v)$ *is an edge of* G, *then* $\mathrm{dist}(v) - \mathrm{dist}(u) \leq 1$.
Not necessarily the case that $\mathrm{dist}(u) - \mathrm{dist}(v) \leq 1$.



BFS with Layers

```
BFSLayers(s):
    Mark all vertices as unvisited and initialize T to be empty
    Mark s as visited and set L_0 = {s}
    i = 0
    while L_i is not empty do
            initialize L_{i+1} to be an empty list
            for each u in L_i do
                for each edge (u, v) ∈ Adj(u) do
                if v is not visited
                        mark v as visited
                        add (u, v) to tree T
                        add v to L_{i+1}
            i = i + 1
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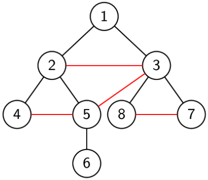
Running time: $O(n+m)$

BFS with Layers: Properties

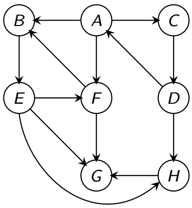


Proposition

1. **BFSLayers**(s) *outputs a* **BFS** *tree*
2. L_i *is the set of vertices at shortest distance exactly* i *from* s
3. *If* G *is undirected, each edge* $e = \{u, v\}$ *is:*
   1. **tree** *edge between two* consecutive *layers*
   2. *non-tree* **forward**/**backward** *edge between two consecutive layers*
   3. *non-tree* **cross-edge** *with both* u, v *in same layer*
   4. $\implies$ *Every edge in the graph is either between two vertices that are either (i) in the same layer, or (ii) in two consecutive layers.*

BFS with Layers: Properties

For directed graphs



Proposition

The following properties hold on termination of **BFSLayers**(s), *if* G *is directed.*
For each edge $e = (u, v)$ *is one of four types:*

1. *a* **tree** *edge between consecutive layers,* $u \in L_i, v \in L_{i+1}$ *for some* $i \geq 0$
2. *a non-tree* **forward** *edge between consecutive layers*
3. *a non-tree* **backward** *edge*
4. *a* **cross-edge** *with both* u, v *in same layer*

Part II

Shortest Paths and Dijkstra's Algorithm

Shortest Path Problems

Shortest Path Problems

Input A (undirected or directed) graph $G = (V, E)$ with edge lengths (or costs). For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

1. Given nodes s, t find shortest path from s to t.
2. Given node s find shortest path from s to all other nodes.
3. Find shortest paths for all pairs of nodes.

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Many applications!

Single-Source Shortest Path Problems

1. Input: A (undirected or directed) graph $G = (V, E)$ with non-negative edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.
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1. Restrict attention to directed graphs
2. Undirected graph problem can be reduced to directed graph problem
   1. Given undirected graph G, create a new directed graph G' by replacing each edge $\{u, v\}$ in G by (u, v) and (v, u) in G'.
   2. set $\ell(u, v) = \ell(v, u) = \ell(\{u, v\})$
   3. Exercise: show reduction works. Relies on non-negativity!

Single-Source Shortest Paths via BFS

Special case: All edge lengths are $\mathbf{1}$.

Single-Source Shortest Paths via BFS

Special case: All edge lengths are 1.

1. Run **BFS**(s) to get shortest path distances from s to all other nodes.
2. $O(m + n)$ time algorithm.

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Can we use **BFS**?

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Can we use **BFS**? Reduce to unit edge-length problem by placing $\ell(e) - 1$ dummy nodes on e

Let $L = \max_e \ell(e)$. New graph has $O(mL)$ edges and $O(mL+n)$ nodes. **BFS** takes $O(mL+n)$ time. Not efficient if L is large.

Towards an algorithm

Why does **BFS** work?

Towards an algorithm

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BFS(s) explores nodes in increasing distance from s

Towards an algorithm

Why does **BFS** work?

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Lemma

Let G be a directed graph with non-negative edge lengths. Let $\mathrm{dist}(s, v)$ denote the shortest path length from s to v. If $s = v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_k$ is a shortest path from s to v_k then for $1 \leq i < k$:

1. *$s = v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_i$ is a shortest path from s to v_i*
2. *$\mathrm{dist}(s, v_i) \leq \mathrm{dist}(s, v_k)$. Relies on non-neg edge lengths.*

Towards an algorithm

Lemma

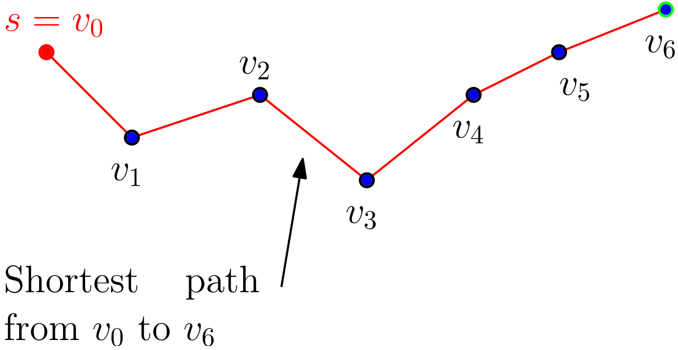
Let G be a directed graph with non-negative edge lengths. Let $\mathrm{dist}(s, v)$ denote the shortest path length from s to v. If $s = v_0 \to v_1 \to \ldots \to v_k$ is a shortest path from s to v_k then for $1 \leq i < k$:

1. $s = v_0 \to v_1 \to \ldots \to v_i$ *is a shortest path from* s *to* v_i
2. $\mathrm{dist}(s, v_i) \leq \mathrm{dist}(s, v_k)$. *Relies on non-neg edge lengths.*

Proof.

If there is a strictly shorter path P' from s to v_i (compared to $s = v_0 \to v_1 \to \ldots \to v_i$), then P' concatenated with $v_i \to v_{i+1} \ldots \to v_k$ gives a strictly shorter path from s to v_k.
For the second part, observe that edge lengths are non-negative. □

A proof by picture



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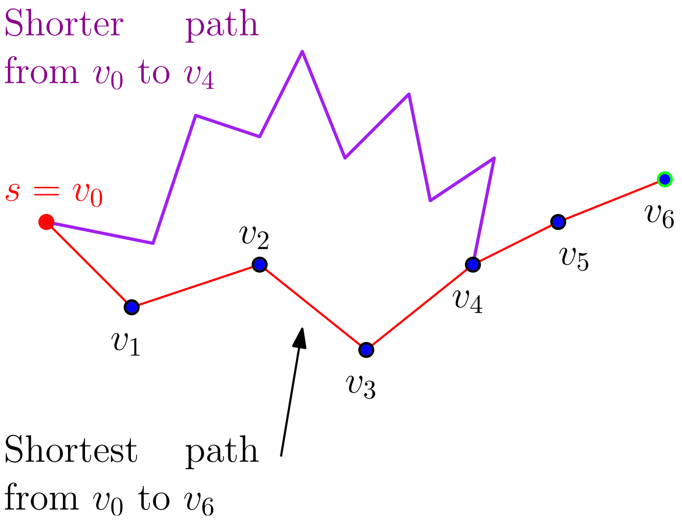

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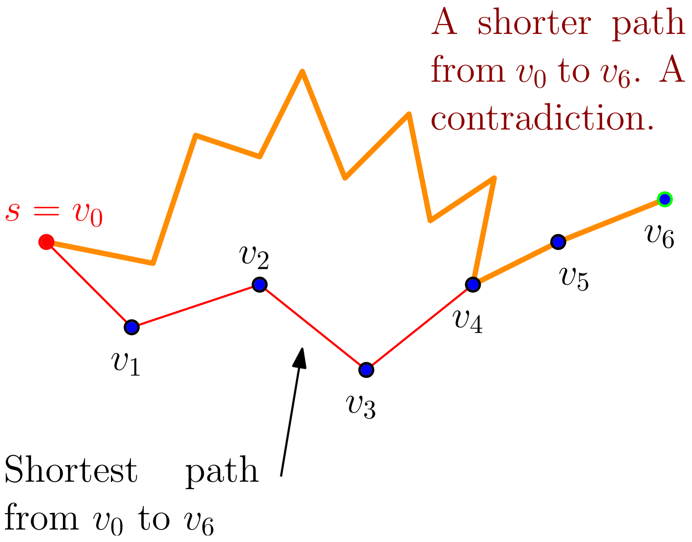

A Basic Strategy

Explore vertices in increasing order of distance from s:
(For simplicity assume that nodes are at different distances from s and that no edge has zero length)

```
Initialize for each node v, dist(s, v) = ∞
Initialize X = {s},
for i = 2 to |V| do
    (* Invariant: X contains the i − 1 closest nodes to s *)
    Among nodes in V − X, find the node v that is the
            i'th closest to s
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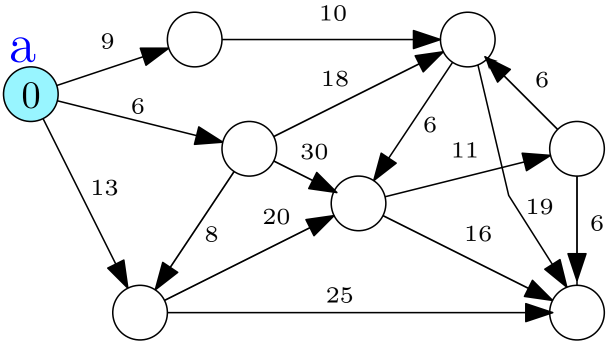
How can we implement the step in the for loop?

Finding the ith closest node repeatedly

An example



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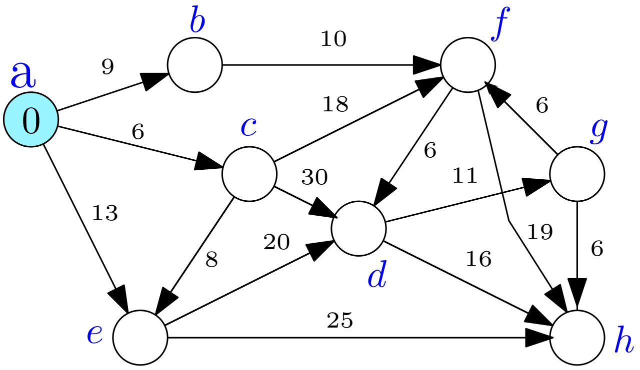

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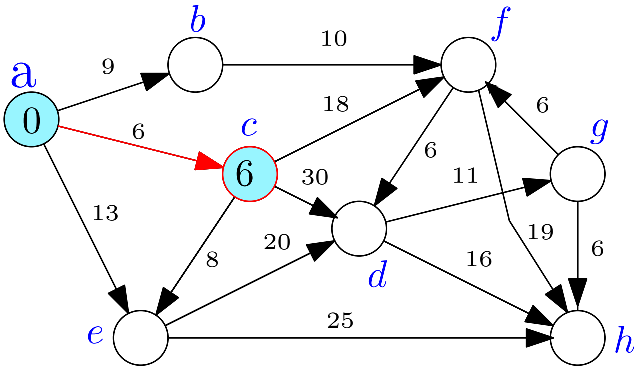

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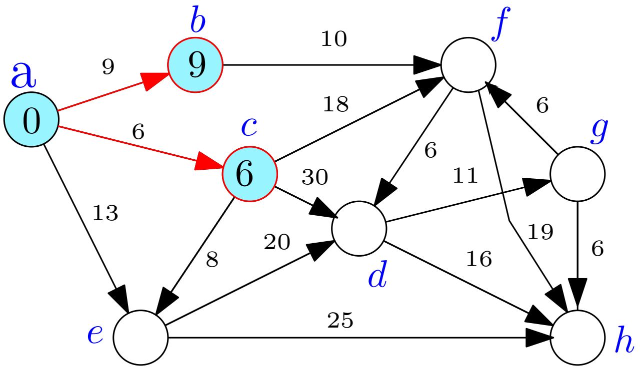

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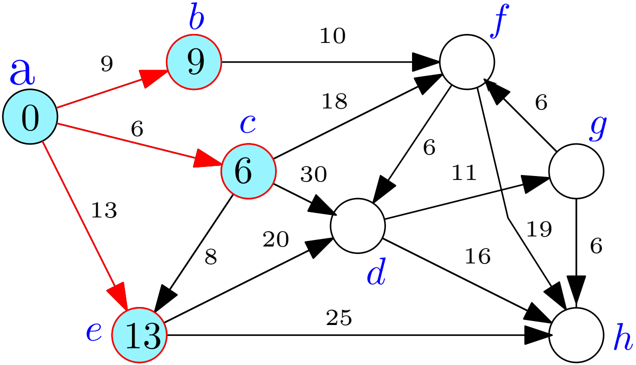

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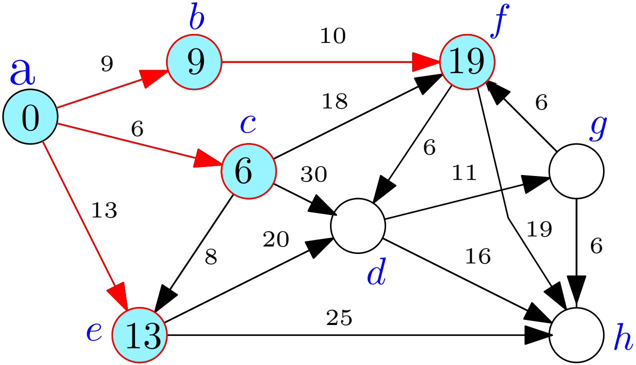

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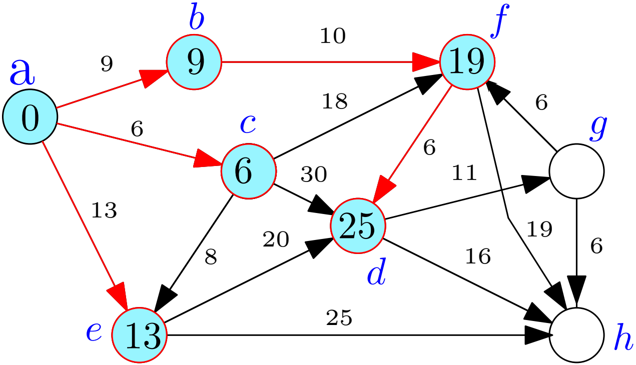

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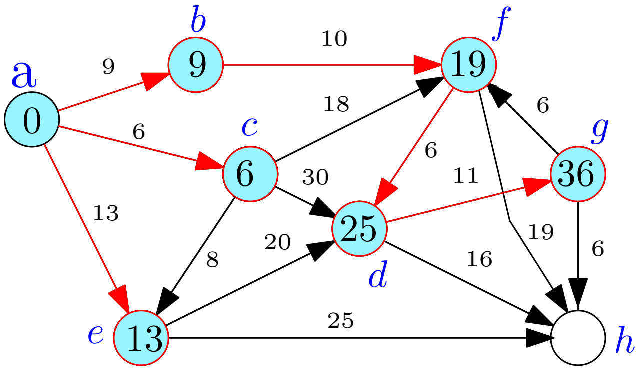

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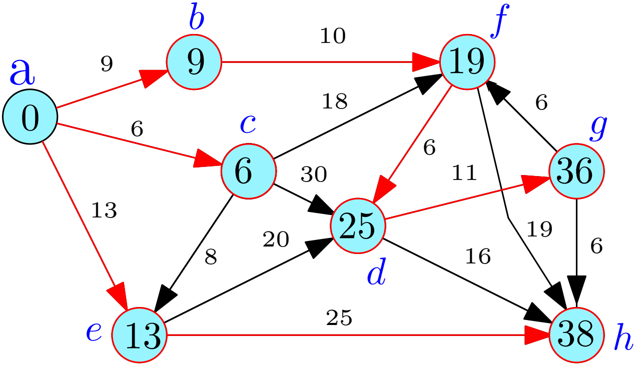

Finding the ith closest node

1. X contains the $i - 1$ closest nodes to s
2. Want to find the ith closest node from $V - X$.

What do we know about the ith closest node?

Claim

If v is the ith closest node with shortest path P from s, then all intermediate nodes of P belong to X.

Finding the ith closest node

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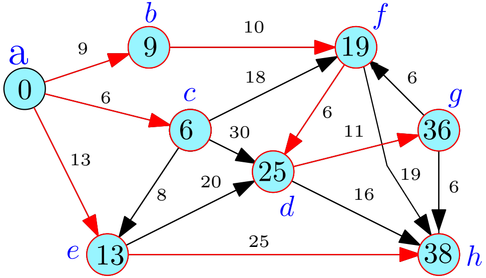
If v is the ith closest node with shortest path P from s, then all intermediate nodes of P belong to X.

Proof.

Every intermediate node of P is closer to s than v (prev Lemma). Since X containts all $(i - 1)$ closest nodes, and v is the ith closest node, the claim follows. $\square$

Finding the ith closest node



Corollary

The ith closest node is adjacent to X.

Finding the ith closest node

1. X contains the $i - 1$ closest nodes to s
2. Want to find the ith closest node from $V - X$. We know:
   - ith closest node, say v, is adjacent to X.
   - all intermediate nodes of the $s \to v$ shortest path are in X.

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1. For each $u \in V - X$ let $P(s, u, X)$ be a shortest path from s to u using only nodes in X as intermediate vertices.
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Observations: for each $u \in V - X$,

1. $\mathrm{dist}(s, u) \leq d'(s, u)$ since we are constraining the paths
2. $d'(s, u) = \min_{t \in X}(\mathrm{dist}(s, t) + \ell(t, u))$ - Why?

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Lemma

If v is the ith closest node to s, then $d'(s, v) = \mathrm{dist}(s, v)$, and $d'(s, v) \leq d'(s, u)$, $\forall u \in V - X$.

Finding the ith closest node

Lemma

Given:

1. X: *Set of* $i - 1$ *closest nodes to* s.
2. $d'(s, u) = \min_{t \in X}(\mathrm{dist}(s, t) + \ell(t, u))$

If v *is an* i*th closest node to* s*, then* $d'(s, v) = \mathrm{dist}(s, v)$.

Proof.

Let v be the ith closest node to s. Then there is a shortest path P from s to v that contains only nodes in X as intermediate nodes (see previous claim). Therefore $d'(s, v) = \mathrm{dist}(s, v)$. □

Finding the ith closest node

Lemma

If v is an ith closest node to s, then $d'(s, v) = \mathrm{dist}(s, v)$.

Corollary

The ith closest node to s is the node $v \in V - X$ such that $d'(s, v) = \min_{u \in V - X} d'(s, u)$.

Proof.

For every node $u \in V - X$, $\mathrm{dist}(s, u) \leq d'(s, u)$ and for the ith closest node v, $d'(s, v) = \mathrm{dist}(s, v)$. Moreover, $\mathrm{dist}(s, v) \leq \mathrm{dist}(s, u)$ for each $u \in V - X$. $\square$

Algorithm

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Initialize for each node v: dist(s, v) = ∞
Initialize X = ∅, d'(s, s) = 0
for i = 1 to |V| do
    (* Invariant: X contains the i − 1 closest nodes to s *)
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     using only X as intermediate nodes*)
    Let v be such that d'(s, v) = min_{u∈V−X} d'(s, u)
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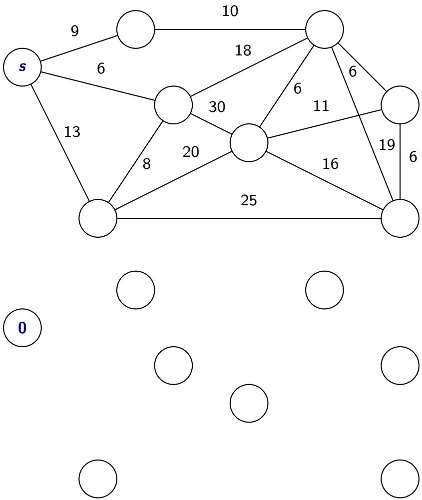
Correctness: By induction on i using previous lemmas.

Running time: $O(n \cdot (n + m))$ time.

1. n outer iterations. In each iteration, $d'(s, u)$ for each u by scanning all edges out of nodes in X; $O(m + n)$ time/iteration.

Example



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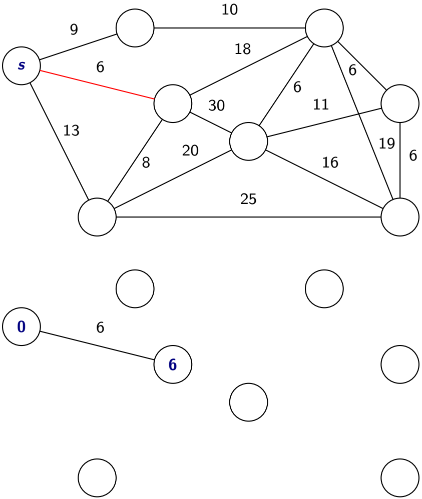

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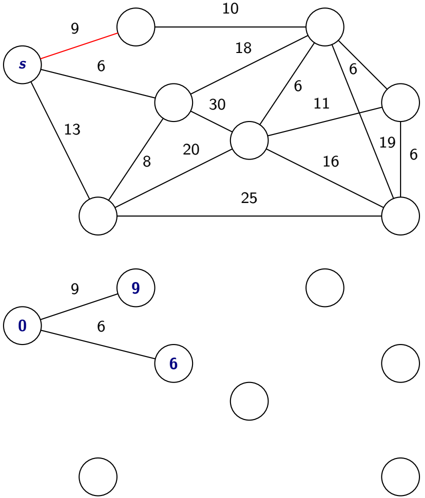

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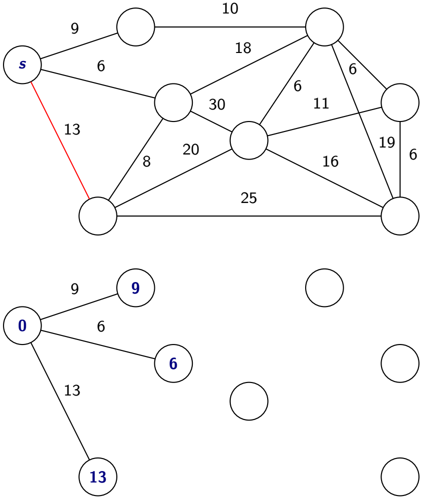

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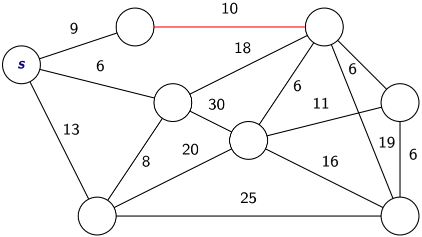

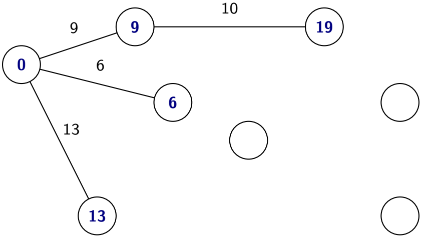

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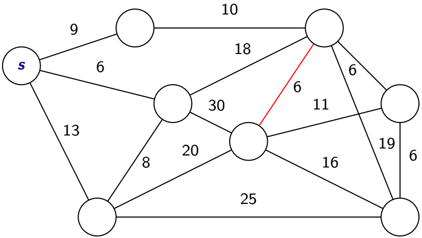

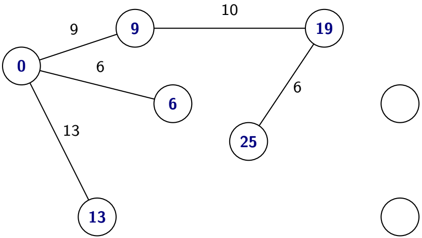

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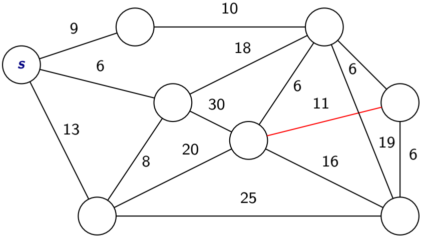

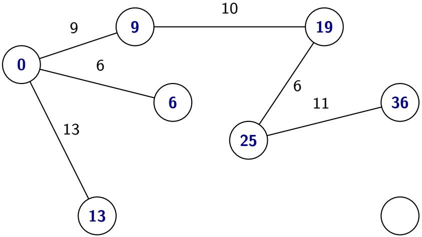

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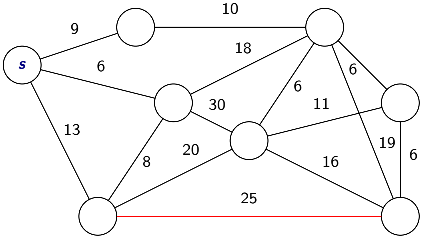

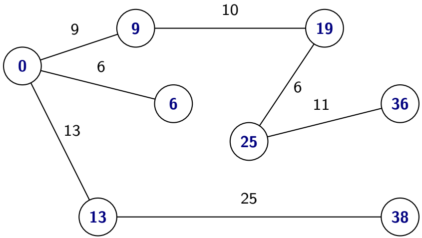

Improved Algorithm

1. Main work is to compute the $d'(s, u)$ values in each iteration
2. $d'(s, u)$ changes from iteration i to $i + 1$ only because of the node v that is added to X in iteration i.

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Running time:

Improved Algorithm

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Running time: $O(m + n^2)$ time.

1. n outer iterations and in each iteration following steps
2. updating $d'(s, u)$ after v is added takes $O(deg(v))$ time so total work is $O(m)$ since a node enters X only once
3. Finding v from $d'(s, u)$ values is $O(n)$ time

Dijkstra's Algorithm

1. eliminate $d'(s, u)$ and let $\mathrm{dist}(s, u)$ maintain it
2. update *dist* values after adding v by scanning edges out of v

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Priority Queues to maintain *dist* values for faster running time

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Priority Queues to maintain *dist* values for faster running time

1. Using heaps and standard priority queues: $O((m + n) \log n)$
2. Using Fibonacci heaps: $O(m + n \log n)$.

Priority Queues

Data structure to store a set S of n elements where each element $v \in S$ has an associated real/integer key $k(v)$ such that the following operations:

1. **makePQ**: create an empty queue.
2. **findMin**: find the minimum key in S.
3. **extractMin**: Remove $v \in S$ with smallest key and return it.
4. **insert**$(v, k(v))$: Add new element v with key $k(v)$ to S.
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6. **decreaseKey**$(v, k'(v))$: *decrease* key of v from $k(v)$ (current key) to $k'(v)$ (new key). Assumption: $k'(v) \leq k(v)$.
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All operations can be performed in $O(\log n)$ time.
decreaseKey is implemented via **delete** and **insert**.

Dijkstra's Algorithm using Priority Queues

```
Q ← makePQ()
insert(Q, (s,0))
for each node u ≠ s do
    insert(Q, (u,∞))
X ← ∅
for i = 1 to |V| do
    (v, dist(s,v)) = extractMin(Q)
    X = X ∪ {v}
    for each u in Adj(v) do
        decreaseKey(Q, (u, min(dist(s,u), dist(s,v) + ℓ(v,u)))).
```

Priority Queue operations:

1. $O(n)$ **insert** operations
2. $O(n)$ **extractMin** operations
3. $O(m)$ **decreaseKey** operations

Implementing Priority Queues via Heaps

Using Heaps

Store elements in a heap based on the key value

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Dijkstra's algorithm can be implemented in $O((n + m) \log n)$ time.

Priority Queues: Fibonacci Heaps/Relaxed Heaps

Fibonacci Heaps

1. **extractMin**, **insert**, **delete**, **meld** in $O(\log n)$ time
2. **decreaseKey** in $O(1)$ *amortized* time:

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1. Dijkstra's algorithm can be implemented in $O(n \log n + m)$ time. If $m = \Omega(n \log n)$, running time is linear in input size.
2. Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps (European Symposium on Algorithms, September 2009!)
3. Active research topic: Paper in 2014 using circuit complexity!

Shortest Path Tree

Dijkstra's algorithm finds the shortest path distances from s to V.
Question: How do we find the paths themselves?

Shortest Path Tree

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```
Q = makePQ()
insert(Q, (s, 0))
prev(s) ← null
for each node u ≠ s do
    insert(Q, (u, ∞) )
    prev(u) ← null

X = ∅
for i = 1 to |V| do
    (v, dist(s, v)) = extractMin(Q)
    X = X ∪ {v}
    for each u in Adj(v) do
        if (dist(s, v) + ℓ(v, u) < dist(s, u)) then
            decreaseKey(Q, (u, dist(s, v) + ℓ(v, u)))
            prev(u) = v
```

Shortest Path Tree

Lemma

The edge set $(\mathrm{prev}(u), u)$ *is a shortest path tree rooted at* s*. For each* u*, the path from* s *to* u *in the tree is a shortest path from* s *to* u*.*

Proof Sketch.

1. The edge set $\{(\mathrm{prev}(u), u) \mid u \in V\}$ induces a directed out-tree rooted at s (Why?)
2. Use induction on $|X|$ to argue that the tree is a shortest path tree for nodes in V.

□

Shortest paths to s

Dijkstra's algorithm gives shortest paths from s to all nodes in V.
How do we find shortest paths from all of V to s?

Shortest paths to s

Dijkstra's algorithm gives shortest paths from s to all nodes in V. How do we find shortest paths from all of V to s?

1. In undirected graphs shortest path from s to u is a shortest path from u to s so there is no need to distinguish.
2. In directed graphs, use Dijkstra's algorithm in G^{rev}!

Shortest paths between sets of nodes

Suppose we are given $S \subset V$ and $T \subset V$. Want to find shortest path from S to T defined as:

$$\mathrm{dist}(S, T) = \min_{s \in S, t \in T} \mathrm{dist}(s, t)$$

How do we find $\mathrm{dist}(S, T)$?

Example Problem

You want to go from your house to a friend's house. Need to pick up some dessert along the way and hence need to stop at one of the many potential stores along the way. How do you calculate the "shortest" trip if you include this stop?

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$O(|X|(m + n \log n))$.

Better solution: Compute shortest path distances from s to every node $v \in V$ with one Dijkstra. Compute from every node $v \in V$ shortest path distance to t with one Dijkstra. $O(m + n \log n)$.