CS/ECE 374 A (Spring 2020)
Old HW1 Problems with Solutions

Problem OLD.1.1: Consider the recurrence

$$T(n) = \begin{cases} T(\lfloor n/3 \rfloor) + T(\lfloor n/4 \rfloor) + T(\lfloor n/5 \rfloor) + T(\lfloor n/6 \rfloor) + n & n \geq 6 \\ 1 & n < 6. \end{cases}$$

Prove by induction that $T(n) = O(n)$.

Solution:

Claim 1. *For $c \geq 20$, and for all $n \geq 1$, we have $T(n) \leq cn$.*

Proof. **Base case.** For $n < 6$ the claim holds for any $c \geq 1$ by definition.

Induction hypothesis. Let $n \geq 6$. Assume that $T(k) \leq ck$ for all $1 \leq k < n$.

Induction step. We need to prove that $T(n) \leq cn$. We know that

$$\begin{aligned} T(n) &= T(\lfloor n/3 \rfloor) + T(\lfloor n/4 \rfloor) + T(\lfloor n/5 \rfloor) + T(\lfloor n/6 \rfloor) + n \\ &\leq c\lfloor n/3 \rfloor + c\lfloor n/4 \rfloor) + c\lfloor n/5 \rfloor) + c\lfloor n/6 \rfloor) + n \qquad \text{(by the induction hypothesis)} \\ &\leq cn/3 + cn/4 + cn/5 + cn/6 + n \\ &\leq \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right)cn + n \;=\; \left(\frac{3}{4} + \frac{1}{5}\right)cn + n \;=\; \left(\frac{19}{20}c + 1\right)n \;\leq\; cn, \end{aligned}$$

provided that

$$\frac{19}{20}c + 1 \leq c \iff 1 \leq \frac{1}{20}c \iff c \geq 20.$$

□

IMPORTANT NOTE: make sure that the "c" in the conclusion from the induction step ($T(n) \leq cn$) is the same as the "c" you start with from the induction hypothesis ($T(k) \leq ck$ for $k < n$). If not (for example, if you could only conclude that $T(n) \leq 1.01cn$), then the whole proof would be incorrect—because the constant factor will "blow up" when we repeat!

This leads to another important piece of advice: don't use big-O notation inside induction proofs!

Problem OLD.1.2: Let $L \subseteq \{0,1\}^*$ be a language defined recursively as follows:

- $\varepsilon \in L$.
- For all $w \in L$ we have $0w1 \in L$.
- For all $x, y \in L$ we have $xy \in L$.
- And these are all the strings that are in L.

Prove, by induction, that for any $w \in L$, and any prefix u of w, we have that $\#_0(u) \geq \#_1(u)$. Here $\#_0(u)$ is the number of 0 appearing in u ($\#_1(u)$ is defined similarly). You can use without proof that $\#_0(xy) = \#_0(x) + \#_0(y)$, for any strings x, y.

Solution:

Proof. The proof is by induction on the length of w.

Base case: If $|w| = 0$ then $w = \varepsilon$, and then $\#_0(w) = 0 \geq \#_1(u) = 0$. Since the only prefix of the empty string is itself, the claim readily follows.

Induction hypothesis: Assume that the claim holds for all strings of length $< n$.

Induction step: We need to prove the claim for a string w of length n. There are two possibilities:

- $w = 0z1$, for some string $z \in L$.
  Let u be any prefix of w. If $u = \varepsilon$ or $u = 0$ then the claim clearly holds for u.
  If $u = w$, then

$$\#_0(u) \;=\; \#_0(w) \;=\; 1 + \#_0(z) + 0 \;\geq\; 1 + \#_1(z) \;=\; \#_1(w) \;=\; \#_1(u),$$

  which implies the claim (we used the induction hypothesis on z, since $z \in L$ and $|z| = |w| - 2 < n$).
  So the remaining case is when $u = 0z'$, where z' is a prefix of z. In this case,

$$\#_0(u) \;=\; \#_0(0z') \;=\; 1 + \#_0(z') \;\geq\; 1 + \#_1(z') \;=\; 1 + \#_1(u) \;>\; \#_1(u),$$

  Again, we used the induction hypothesis on z, since $z \in L$, z' is a prefix of z, and z strictly shorter than w. This implies the claim.
- $w = xy$, for some strings $x, y \in L$, such that $|x|, |y| > 0$.
  Let u be a prefix of w. If u is a prefix of x, then the claim holds readily by induction. The remaining case is when $u = xz$, for some z which is prefix of y. Here,

$$\#_0(u) \;=\; \#_0(xz) \;=\; \#_0(x) + \#_0(z) \;\geq\; \#_1(x) + \#_1(z) \;=\; \#_1(u),$$

  by using the induction hypothesis on x (which is a prefix of itself), and on z (which is a prefix of y), noting that both x and y are strictly shorter than w.

□

Problem OLD.1.3. Recall that the *reversal* w^R of a string w is defined recursively as follows:

$$w^R := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ x^R \bullet a & \text{if } w = a \cdot x \end{cases}$$

A *palindrome* is any string that is equal to its reversal, like *AMANAPLANACANALPANAMA*, *RACECAR*, *POOP*, *I*, and the empty string.

(a) Give a recursive definition of a palindrome over the alphabet Σ.

(b) Prove $w = w^R$ for every palindrome w (according to your recursive definition).

(c) Prove that every string w such that $w = w^R$ is a palindrome (according to your recursive definition).

In parts (b) and (c), you may assume without proof that $(x \cdot y)^R = y^R \bullet x^R$ and $(x^R)^R = x$ for all strings x and y.

Solution:

(a) A string $w \in \Sigma^*$ is a palindrome if and only if either

- $w = \varepsilon$, or
- $w = a$ for some symbol $a \in \Sigma$, or
- $w = axa$ for some symbol $a \in \Sigma$ and some *palindrome* $x \in \Sigma^*$.

(b) Let w be an arbitrary palindrome.
Assume that $x = x^R$ for every palindrome x such that $|x| < |w|$.
There are three cases to consider (mirroring the three cases in the definition):

- If $w = \varepsilon$, then $w^R = \varepsilon$ by definition, so $w = w^R$.
- If $w = a$ for some symbol $a \in \Sigma$, then $w^R = a$ by definition, so $w = w^R$.
- Suppose $w = axa$ for some symbol $a \in \Sigma$ and some palindrome $x \in P$. Then

$$\begin{aligned}
w^R &= (a \cdot x \bullet a)^R \\
&= (x \bullet a)^R \bullet a && \text{by definition of reversal} \\
&= a^R \bullet x^R \bullet a && \text{You said we could assume this.} \\
&= a \bullet x^R \bullet a && \text{by definition of reversal} \\
&= a \bullet x \bullet a && \text{by the inductive hypothesis} \\
&= w && \text{by assumption}
\end{aligned}$$

In all three cases, we conclude that $w = w^R$.

(c) Let w be an arbitrary string such that $w = w^R$.
Assume that every string x such that $|x| < |w|$ and $x = x^R$ is a palindrome.
There are three cases to consider (mirroring the definition of "palindrome"):

- If $w = \varepsilon$, then w is a palindrome by definition.
- If $w = a$ for some symbol $a \in \Sigma$, then w is a palindrome by definition.
- Otherwise, we have $w = ax$ for some symbol a and some *non-empty* string x.
  The definition of reversal implies that $w^R = (ax)^R = x^R a$.
  Because x is non-empty, its reversal x^R is also non-empty.
  Thus, $x^R = by$ for some symbol b and some string y.
  It follows that $w^R = bya$, and therefore $w = (w^R)^R = (bya)^R = ay^R b$.

  [At this point, we need to prove that $a = b$ and that y is a palindrome.]

  Our assumption that $w = w^R$ implies that $bya = ay^R b$.
  The recursive definition of string equality immediately implies $a = b$.

Because $a = b$, we have $w = ay^R a$ and $w^R = aya$.
The recursive definition of string equality implies $y^R a = ya$.
It immediately follows that $(y^R a)^R = (ya)^R$.
Known properties of reversal imply $(y^R a)^R = a(y^R)^R = ay$ and $(ya)^R = ay^R$.
It follows that $ay^R = ay$, and therefore $y = y^R$.
The inductive hypothesis now implies that y is a palindrome.

We conclude that w is a palindrome by definition.

In all three cases, we conclude that w is a palindrome.