# CS/ECE 374 A (Spring 2020)
# Homework 11 (due Apr 30 Thursday at 10am)

**Instructions:** As in previous homeworks. See Old HW 11 for tips and examples on how to write NP-completeness proofs.

**Problem 11.1:** Given an undirected graph $G$, we want to decide whether $G$ contains a spanning tree where every node has degree at most 4. Prove that this problem is NP-complete.

[Hint: you may assume that the Hamiltonian Path problem is NP-complete.]

**Problem 11.2:** We need to schedule final exams for $N$ classes. We want to minimize the number of days, but don't want any students to take more than 2 exams on a single day.

One way to formulate this problem is as follows: There are $M$ students, and for each $j = 1, \ldots, M$, we are given a set $S_j \subseteq \{1, \ldots, N\}$ of the classes that student $j$ is taking. We are also given an integer $D$. We want to decide whether there exists a function $f : \{1, \ldots, N\} \to \{1, \ldots, D\}$ such that for every $j$ and $k$, the number of elements in $\{x \in S_j \mid f(x) = k\}$ is at most 2.

Prove that this problem is NP-complete.

[Hint: you may assume that 3-Coloring is NP-complete. Observe that if we create 4 copies of a vertex $v$ and $D = 3$, then two copies of $v$ must have the same $f$ value. For each edge $uv$, create a constant number of sets (of size 3 or 4)...]

**Problem 11.3:** Consider the following version of the Crossword-Puzzle problem:

> *Input*: $A_1, \ldots, A_m, B_1, \ldots, B_n$, where each $A_i$ is a finite set of length-$n$ strings and each $B_j$ is a finite set of length-$m$ strings, over a finite alphabet $\Sigma$.
>
> *Output*: "yes" iff there exists an $m \times n$ table $T$ of symbols such that for each $i = 1, \ldots, m$, the $i$-th row of $T$ is a string in the set $A_i$, and for each $j = 1, \ldots, n$, the $j$-th column of $T$ is a string in the set $B_j$.

Example: on the input $A_1 = \{\texttt{CAT}, \texttt{DOG}\}$, $A_2 = \{\texttt{CAT}, \texttt{APE}, \texttt{AGO}\}$, $A_3 = \{\texttt{BAD}, \texttt{BEE}\}$, $B_1 = \{\texttt{CAB}, \texttt{DAB}\}$, $B_2 = \{\texttt{APE}, \texttt{EGG}\}$, and $B_3 = \{\texttt{GOD}, \texttt{TEE}\}$, the answer is yes, with the following solution $T$:

```
CAT
APE
BEE
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Prove that Crossword-Puzzle is NP-complete.

[Hint: reduce from 3SAT. Given a 3CNF formula $F$ with variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$, let $b_j$ be the length-$m$ binary string (over $\Sigma = \{0, 1\}$) such that the $i$-th bit is 1 iff $x_j$ appears in $C_i$, and let $b'_j$ be the length-$m$ binary string such that the $i$-th bit is 1 iff $\overline{x_j}$ appears in $C_i$. Define $B_j = \{b_j, b'_j\}$, which contains 2 strings. Now define $A_i$ to contain 7 appropriately chosen strings...]