Dynamic Programming

Lecture 13

Thursday, February 28, 2019

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Part I

Recursion and Memoization

Fibonacci Numbers

Fibonacci numbers defined by recurrence:

$$F(n) = F(n-1) + F(n-2) \text{ and } F(0) = 0, F(1) = 1.$$

These numbers have many interesting and amazing properties.
A journal *The Fibonacci Quarterly*!

1. $F(n) = (\phi^n - (1-\phi)^n)/\sqrt{5}$ where ϕ is the golden ratio $(1+\sqrt{5})/2 \simeq 1.618$.
2. $\lim_{n\to\infty} F(n+1)/F(n) = \phi$

How many bits?

Consider the nth Fibonacci number $F(n)$. Writing the number $F(n)$ in base 2 requires

- $\Theta(n^2)$ bits.
- $\Theta(n)$ bits.
- $\Theta(\log n)$ bits.
- $\Theta(\log \log n)$ bits.

Recursive Algorithm for Fibonacci Numbers

Question: Given n, compute $F(n)$.

```
Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else
        return Fib(n − 1) + Fib(n − 2)
```

Running time? Let $T(n)$ be the number of additions in Fib(n).

$$T(n) = T(n-1) + T(n-2) + 1 \text{ and } T(0) = T(1) = 0$$

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$$T(n) = T(n-1) + T(n-2) + 1 \text{ and } T(0) = T(1) = 0$$

Roughly same as $F(n)$

$$T(n) = \Theta(\phi^n)$$

The number of additions is exponential in n. Can we do better?

Recursion tree for the Recursive Fibonacci

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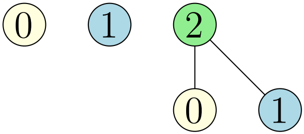

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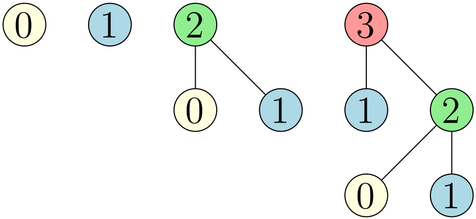

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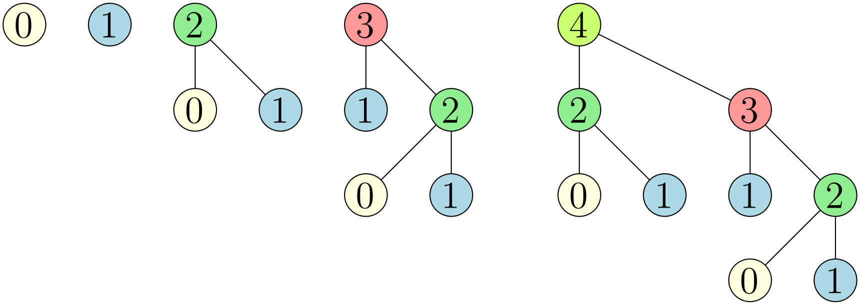

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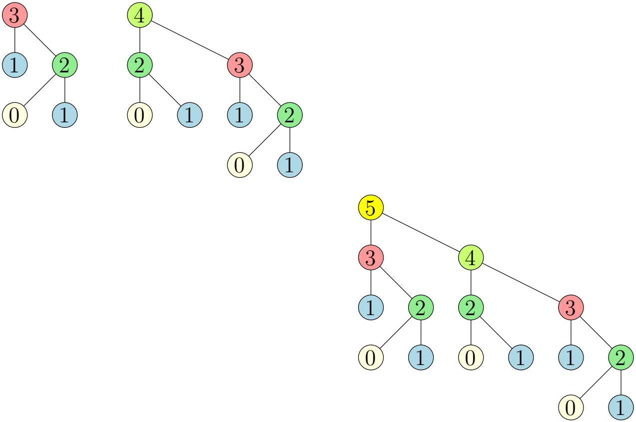

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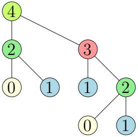

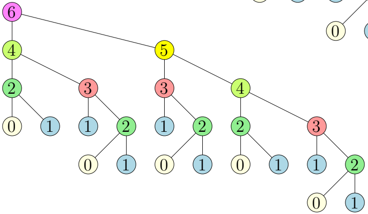

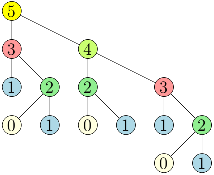

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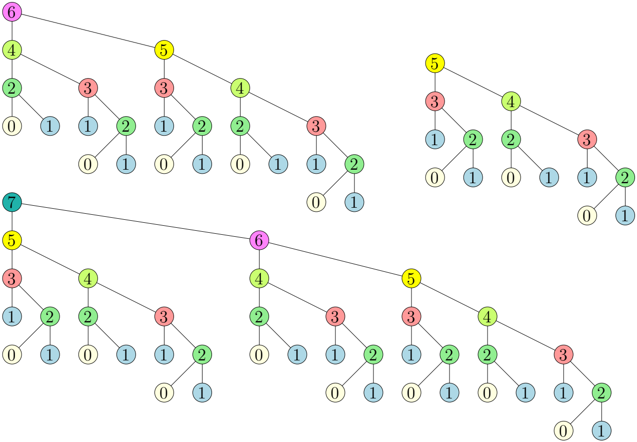

An iterative algorithm for Fibonacci numbers

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FibIter(n):
    if (n = 0) then
        return 0
    if (n = 1) then
        return 1
    F[0] = 0
    F[1] = 1
    for i = 2 to n do
        F[i] = F[i − 1] + F[i − 2]
    return F[n]
```

What is the running time of the algorithm? $O(n)$ additions.

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What is the running time of the algorithm? $O(n)$ additions.

What is the difference?

1. Recursive algorithm is computing the same numbers again and again.
2. Iterative algorithm is storing computed values and building bottom up the final value. Memoization.

Dynamic Programming:

Finding a recursion that can be *effectively/efficiently* memoized.

Leads to polynomial time algorithm if number of sub-problems is polynomial in input size.

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Automatic Memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

```
Fib(n):
    if (n = 0)
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    if (Fib(n) was previously computed)
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        return Fib(n − 1) + Fib(n − 2)
```

How do we keep track of previously computed values?

Two methods: explicitly and implicitly (via data structure)

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Automatic implicit memoization

Initialize a (dynamic) dictionary data structure D to empty

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Fib(n):
    if (n = 0)
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    if (n is already in D)
        return value stored with n in D
    val ⇐ Fib(n − 1) + Fib(n − 2)
    Store (n, val) in D
    return val
```

Use hash-table or a map to remember which values were already computed.

Automatic explicit memoization

1. Initialize table/array M of size n: $M[i] = -1$ for $i = 0, \ldots, n$.
2. Resulting code:

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Fib(n):
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3. Need to know upfront the number of subproblems to allocate memory.

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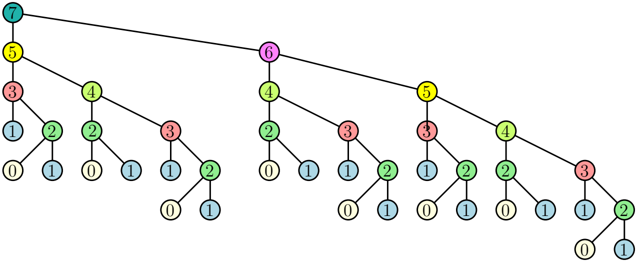

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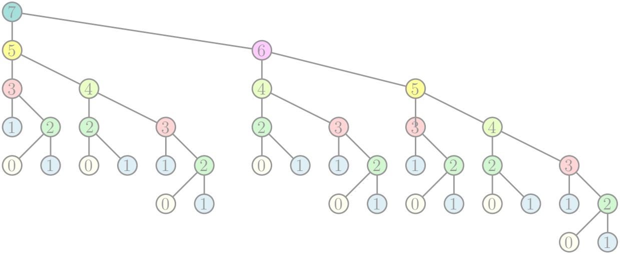

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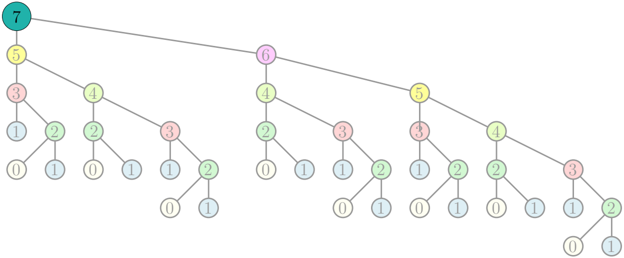

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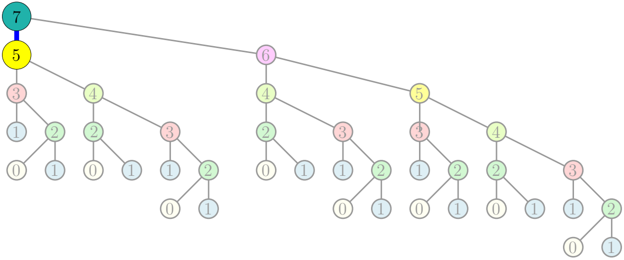

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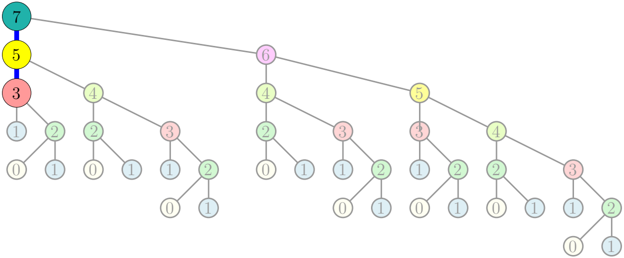

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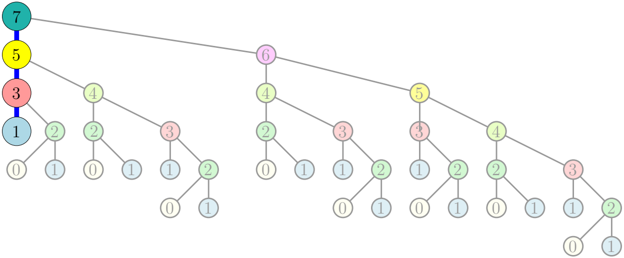

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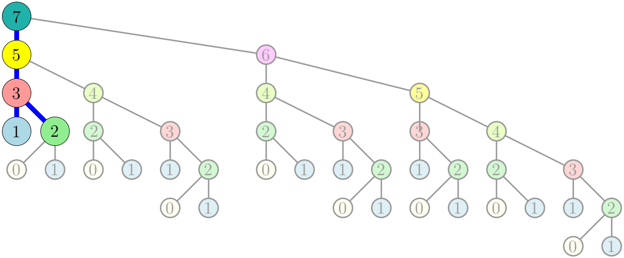

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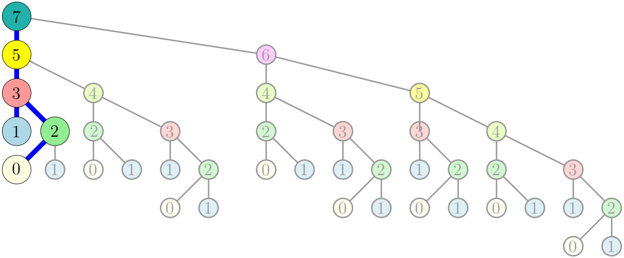

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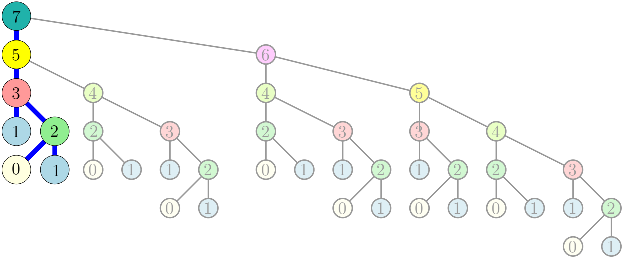

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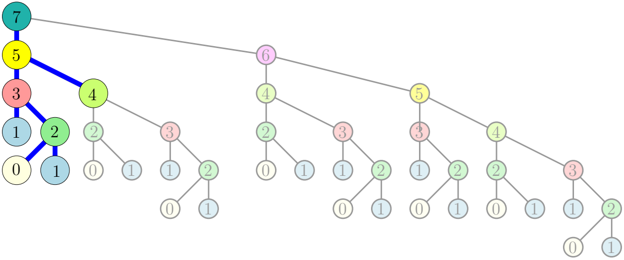

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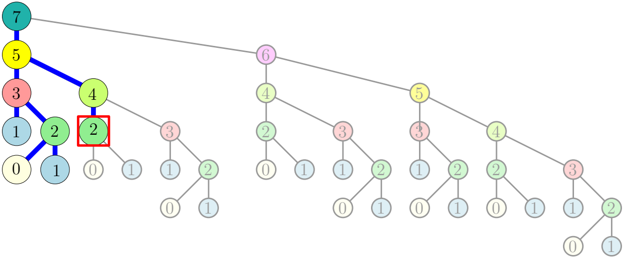

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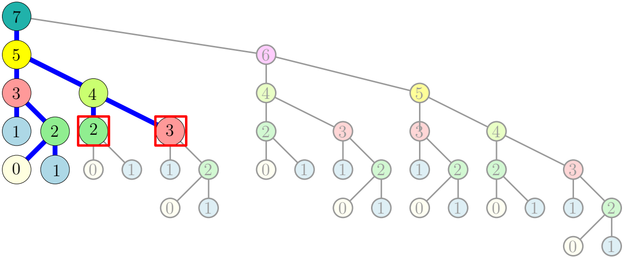

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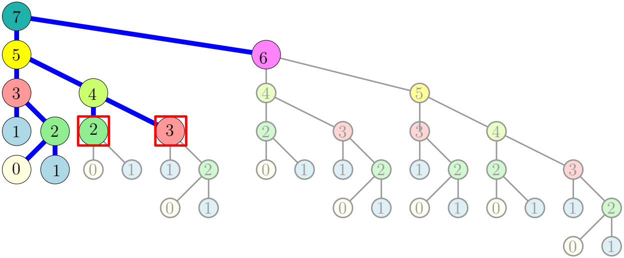

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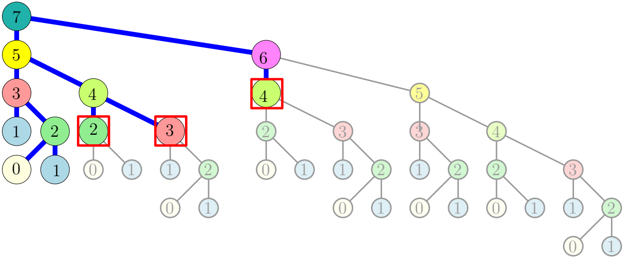

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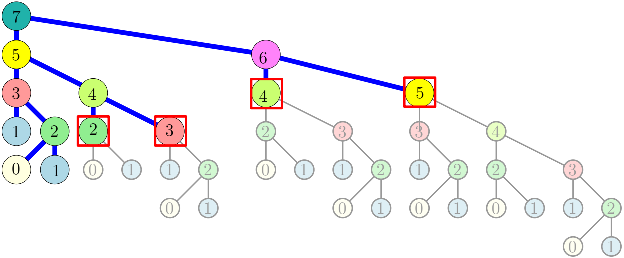

Automatic Memoization

1. Recursive version:

```
f(x_1, x_2, ..., x_d):
        CODE
```

2. Recursive version with memoization:

```
g(x_1, x_2, ..., x_d):
        if f already computed for (x_1, x_2, ..., x_d) then
            return value already computed
        NEW_CODE
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3. NEW_CODE:
   1. Replaces any "**return** α" with
   2. Remember "$f(x_1, \ldots, x_d) = \alpha$"; **return** α.

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Explicit vs Implicit Memoization

1. Explicit memoization (iterative algorithm) preferred:
   1. analyze problem ahead of time
   2. Allows for efficient memory allocation and access.
2. Implicit (automatic) memoization:
   1. problem structure or algorithm is not well understood.
   2. Need to pay overhead of data-structure.
   3. Functional languages (e.g., LISP) automatically do memoization, usually via hashing based dictionaries.

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Initialize table/array M of size n such that $M[i] = -1$ for $i = 0, \ldots, n$.

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Fib(n):
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    if (M[n] ≠ −1) (* M[n] has stored value of Fib(n) *)
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    M[n] ⇐ Fib(n − 1) + Fib(n − 2)
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To allocate memory need to know upfront the number of subproblems for a given input size n

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        val ⇐ Fib(n − 1) + Fib(n − 2)
    Store (n, val) in D
    return val
```

Explicit vs Implicit Memoization

1. Explicit memoization or iterative algorithm preferred if one can analyze problem ahead of time. Allows for efficient memory allocation and access.
2. Implicit and automatic memoization used when problem structure or algorithm is either not well understood or in fact unknown to the underlying system.
    1. Need to pay overhead of data-structure.
    2. Functional languages such as LISP automatically do memoization, usually via hashing based dictionaries.

How many distinct calls?

```
binom(t, b)    // computes (t choose b)
    if t = 0 then return 0
    if b = t or b = 0 then return 1
    return binom(t − 1, b − 1) + binom(t − 1, b).
```

How many distinct calls does **binom**$(n, \lfloor n/2 \rfloor)$ makes during its recursive execution?

- $\Theta(1)$.
- $\Theta(n)$.
- $\Theta(n \log n)$.
- $\Theta(n^2)$.
- $\Theta\left(\binom{n}{\lfloor n/2 \rfloor}\right)$.

That is, if the algorithm calls recursively **binom**$(17, 5)$ about 5000 times during the computation, we count this is a single distinct call.

Running time of memoized binom?

```
D: Initially an empty dictionary.
binomM(t, b)    // computes (t choose b)
    if b = t then return 1
    if b = 0 then return 0
    if D[t, b] is defined then return D[t, b]
    D[t, b] ⇐ binomM(t − 1, b − 1) + binomM(t − 1, b).
    return D[t, b]
```

Assuming that every arithmetic operation takes $O(1)$ time, What is the running time of **binomM**$(n, \lfloor n/2 \rfloor)$?

- $\Theta(1)$.
- $\Theta(n)$.
- $\Theta(n^2)$.
- $\Theta\left(n^3\right)$.
- $\Theta\left(\binom{n}{\lfloor n/2 \rfloor}\right)$.

Back to Fibonacci Numbers

Is the iterative algorithm a *polynomial* time algorithm? Does it take $O(n)$ time?

1. input is n and hence input size is $\Theta(\log n)$
2. output is $F(n)$ and output size is $\Theta(n)$. Why?
3. Hence output size is exponential in input size so no polynomial time algorithm possible!
4. Running time of iterative algorithm: $\Theta(n)$ additions but number sizes are $O(n)$ bits long! Hence total time is $O(n^2)$, in fact $\Theta(n^2)$. Why?

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Saving space. Do we need an array of n numbers? Not really.

```
FibIter(n):
    if (n = 0) then
        return 0
    if (n = 1) then
        return 1
    prev2 = 0
    prev1 = 1
    for i = 2 to n do
        temp = prev1 + prev2
        prev2 = prev1
        prev1 = temp

    return prev1
```

Part II

Dynamic programming

Dynamic Programming

Dynamic Programming is smart recursion plus memoization

Question: Suppose we have a recursive program $foo(x)$ that takes an input x.

- On input of size n the number of *distinct* sub-problems that $foo(x)$ generates is at most $A(n)$
- $foo(x)$ spends at most $B(n)$ time *not counting* the time for its recursive calls.

Suppose we*memoize* the recursion.

Assumption: Storing and retrieving solutions to pre-computed problems takes $O(1)$ time.

Q: What is an upper bound on the running time of *memoized* version of $foo(x)$ if $|x| = n$? $O(A(n)B(n))$.

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- On input of size n the number of *distinct* sub-problems that $foo(x)$ generates is at most $A(n)$
- $foo(x)$ spends at most $B(n)$ time *not counting* the time for its recursive calls.

Suppose we*memoize* the recursion.

Assumption: Storing and retrieving solutions to pre-computed problems takes $O(1)$ time.

Q: What is an upper bound on the running time of *memoized* version of $foo(x)$ if $|x| = n$? $O(A(n)B(n))$.

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Part III

Checking if a string is in L^*

Problem

Input: A string $w \in \Sigma^*$ and access to a language $L \subseteq \Sigma^*$ via function **IsStrInL**$(string\ x)$ that decides whether x is in L

Goal: Decide if $w \in L^*$ using **IsStrInL**$(string\ x)$ as a black box sub-routine

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Input A string $w \in \Sigma^*$ and access to a language $L \subseteq \Sigma^*$ via function **IsStrInL**$(string\ \ x)$ that decides whether x is in L

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Input A string $w \in \Sigma^*$ and access to a language $L \subseteq \Sigma^*$ via function **IsStrInL**(*string* x) that decides whether x is in L

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Goal Decide if using **IsStrInL**$(string \;\; x)$ as a black box sub-routine

Example

Suppose L is *English* and we have a procedure to check whether a string/word is in the *English* dictionary.

- Is the string "isthisanenglishsentence" in *English**?
- Is "stampstamp" in *English**?
- Is "zibzzzad" in *English**?

Recursive Solution

When is $w \in L^*$?

a $w \in L^*$ if $w \in L$ or if $w = uv$ where $u \in L$ and $v \in L^*$, $|u| \geq 1$

Assume w is stored in array $A[1..n]$

```
IsStringinLstar(A[1..n]):
    If (n = 0) Output YES
    If (IsStrInL(A[1..n]))
        Output YES
    Else
        For (i = 1 to n − 1) do
            If (IsStrInL(A[1..i]) and IsStrInLstar(A[i + 1..n]))
                Output YES

    Output NO
```

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Question: How many distinct sub-problems does IsStrInLstar($A[1..n]$) generate? $O(n)$

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Example

Consider string ***samiam***

Naming subproblems and recursive equation

After seeing that number of subproblems is $O(n)$ we name them to help us understand the structure better.

ISL(i): a boolean which is 1 if $A[i..n]$ is in L^*, 0 otherwise

Base case: **ISL**$(n+1) = 1$ interpreting $A[n+1..n]$ as ϵ

Recursive relation:

- **ISL**$(i) = 1$ if $\exists i < j \leq n+1$ s.t **ISL**(j) and **IsStrInL**$(A[i..(j-1])$
- **ISL**$(i) = 0$ otherwise

Output: **ISL**(1)

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Removing recursion to obtain iterative algorithm

Typically, after finding a dynamic programming recursion, we often convert the recursive algorithm into an *iterative* algorithm via *explicit memoization* and *bottom up* computation.

Why? Mainly for further optimization of running time and space.

How?

- First, allocate a data structure (usually an array or a multi-dimensional array that can hold values for each of the subproblems)
- Figure out a way to order the computation of the sub-problems starting from the base case.

Caveat: Dynamic programming is not about filling tables. It is about finding a smart recursion. First, find the correct recursion.

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Iterative Algorithm

```
IsStringinLstar-Iterative(A[1..n]):
    boolean ISL[1..(n + 1)]
    ISL[n + 1] = TRUE
    for (i = n down to 1)
        ISL[i] = FALSE
        for (j = i + 1 to n + 1)
            If (ISL[j] and IsStrInL(A[i..j − 1]))
                ISL[i] = TRUE
                Break

    If (ISL[1] = 1) Output YES
    Else Output NO
```

- **Running time:** $O(n^2)$ (assuming call to **IsStrInL** is $O(1)$ time)
- **Space:** $O(n)$

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Example

Consider string ***samiam***

Part IV

Longest Increasing Subsequence

13.2: Longest Increasing Subsequence

Sequences

Definition

Sequence: an ordered list $a_1, a_2, \ldots, a_n$. **Length** of a sequence is number of elements in the list.

Definition

$a_{i_1}, \ldots, a_{i_k}$ is a **subsequence** of $a_1, \ldots, a_n$ if
$1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

Definition

A sequence is **increasing** if $a_1 < a_2 < \ldots < a_n$. It is **non-decreasing** if $a_1 \leq a_2 \leq \ldots \leq a_n$. Similarly **decreasing** and **non-increasing**.

Example

1. Sequence: $6, 3, 5, 2, 7, 8, 1, 9$
2. Subsequence of above sequence: $5, 2, 1$
3. Increasing sequence: $3, 5, 9, 17, 54$
4. Decreasing sequence: $34, 21, 7, 5, 1$
5. Increasing subsequence of the first sequence: $2, 7, 9$.

Longest Increasing Subsequence Problem

Input A sequence of numbers $a_1, a_2, \ldots, a_n$

Goal Find an **increasing subsequence** $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

Example

1. Sequence: 6, 3, 5, 2, 7, 8, 1
2. Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
3. Longest increasing subsequence: 3, 5, 7, 8

Longest Increasing Subsequence Problem

Input A sequence of numbers $a_1, a_2, \ldots, a_n$

Goal Find an **increasing subsequence** $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

Example

1. Sequence: 6, 3, 5, 2, 7, 8, 1
2. Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
3. Longest increasing subsequence: 3, 5, 7, 8

Recursive Approach: Take 1

LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

$\text{LIS}(A[1..n])$:

1. Case 1: Does not contain $A[n]$ in which case $\text{LIS}(A[1..n]) = \text{LIS}(A[1..(n-1)])$
2. Case 2: contains $A[n]$ in which case $\text{LIS}(A[1..n])$ is not so clear.

Observation

For second case we want to find a subsequence in $A[1..(n-1)]$ that is restricted to numbers less than $A[n]$. This suggests that a more general problem is **LIS_smaller**$(A[1..n], x)$ *which gives the longest increasing subsequence in A where each number in the sequence is less than x.*

Recursive Approach: Take 1

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Recursive Approach

$LIS(A[1..n])$: the length of longest increasing subsequence in A

LIS_smaller$(A[1..n], x)$: length of longest increasing subsequence in $A[1..n]$ with all numbers in subsequence less than x

```
LIS_smaller(A[1..n], x):
    if (n = 0) then return 0
    m = LIS_smaller(A[1..(n − 1)], x)
    if (A[n] < x) then
        m = max(m, 1 + LIS_smaller(A[1..(n − 1)], A[n]))
    Output m
```

```
LIS(A[1..n]):
    return LIS_smaller(A[1..n], ∞)
```

Example

Sequence: $A[1..7] = 6, 3, 5, 2, 7, 8, 1$

Recursive Approach

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- How many distinct sub-problems will **LIS_smaller**$(A[1..n], \infty)$ generate? $O(n^2)$
- What is the running time if we memoize recursion? $O(n^2)$ since each call takes $O(1)$ time to assemble the answers from to recursive calls and no other computation.
- How much space for memoization? $O(n^2)$

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Naming subproblems and recursive equation

After seeing that number of subproblems is $O(n^2)$ we name them to help us understand the structure better. For notational ease we add ∞ at end of array (in position $n+1$)

$\mathrm{LIS}(i,j)$: length of longest increasing sequence in $A[1..i]$ among numbers less than $A[j]$ (defined only for $i < j$)

Base case: $\mathrm{LIS}(0,j) = 0$ for $1 \leq j \leq n+1$
Recursive relation:

- $\mathrm{LIS}(i,j) = \mathrm{LIS}(i-1,j)$ if $A[i] > A[j]$
- $\mathrm{LIS}(i,j) = \max\{LIS(i-1,j), 1 + LIS(i-1,i)\}$ if $A[i] \leq A[j]$

Output: $\mathrm{LIS}(n, n+1)$

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Output: $\mathrm{LIS}(n, n+1)$

Iterative algorithm

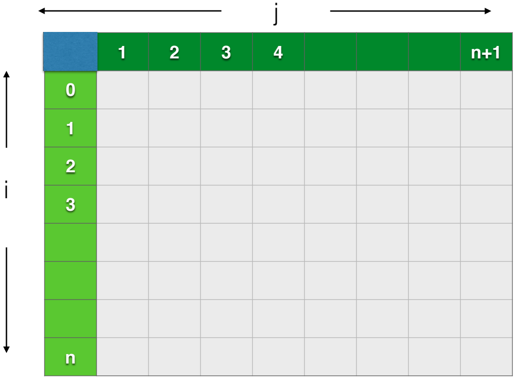
```
LIS-Iterative(A[1..n]):
    A[n + 1] = ∞
    int LIS[0..n, 1..n + 1]
    for (j = 1 to n + 1) do
        LIS[0, j] = 0

    for (i = 1 to n) do
        for (j = i + 1 to n)
            If (A[i] > A[j]) LIS[i, j] = LIS[i − 1, j]
            Else LIS[i, j] = max{LIS[i − 1, j], 1 + LIS[i − 1, i]}

    Return LIS[n, n + 1]
```

Running time: $O(n^2)$
Space: $O(n^2)$

How to order bottom up computation?



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- $\mathrm{LIS}(i, j) = \max\{LIS(i - 1, j), 1 + LIS(i - 1, i)\}$ if $A[i] \leq A[j]$

How to order bottom up computation?

Sequence: $A[1..7] = 6, 3, 5, 2, 7, 8, 1$

j

i

	1	2	3	4				n+1
0								
1								
2								
3								
n								

Two comments

Question: Can we compute an optimum solution and not just its value?

Yes! See notes.

Question: Is there a faster algorithm for LIS? Yes! Using a different recursion and optimizing one can obtain an $O(n \log n)$ time and $O(n)$ space algorithm. $O(n \log n)$ time is not obvious. Depends on improving time by using data structures on top of dynamic programming.

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Dynamic Programming

1. Find a "smart" recursion for the problem in which the number of distinct subproblems is small; polynomial in the original problem size.
2. Estimate the number of subproblems, the time to evaluate each subproblem and the space needed to store the value. This gives an upper bound on the total running time if we use automatic memoization.
3. Eliminate recursion and find an iterative algorithm to compute the problems bottom up by storing the intermediate values in an appropriate data structure; need to find the right way or order the subproblem evaluation. This leads to an explicit algorithm.
4. Optimize the resulting algorithm further

Part V

Some experiments with memoization

Edit distance: different memoizations

Input size	Running time in seconds		
n	DP	Partial	Implicit memoization
1,250	**0.01**	**0.04**	**0.20**
2,500	**0.04**	**0.15**	**0.84**
5,000	**0.18**	**0.64**	**3.73**
10,000	**0.72**	**2.50**	**15.05**
20,000	**2.88**	**9.91**	**55.35**
40,000	**12.00**	**40.00**	out of memory

For the input n, two random strings of length n were generated, and their distance computed using edit distance.
Note, that edit-distance is simple enough to that DP gets very good performance. For more complicated problems, the advantage of DP would probably be much smaller.
The asymptotic running time here is $\Theta(n^2)$.

Edit distance: different memoizations

More details

1. The implementation was done in C++, using -O9 in compilation.
2. DP = Dynamic Programming = iterative implementation using arrays.
3. Partial memoization = Still uses recursive code, but remembers the results in tables that are managed directly by the code.
4. Implicit memoization = implemented using the standard `unordered_map`.

Edit distance: different memoizations

Conclusions

1. If you are in interview setup, you should probably solve the problem using DP. That what you would be expected to do.
2. Otherwise, I would probably implement partial memoization – it still has the simplicity of the recursive solution, while having a decent performance. If I really care about performance I would implement the DP.
3. Using implicit memoization probably makes sense only if running time is not really an issue.