

In this note, we show that any DFA can be converted into a regular expression. Our construction would work by allowing regular expressions to be written on the edges of the DFA, and then showing how one can remove states from this generalized automata (getting a new equivalent automata with the fewer states). In the end of this state removal process, we will remain with a generalized automata with a single initial state and a single accepting state, and it would be then easy to convert it into a single regular expression.

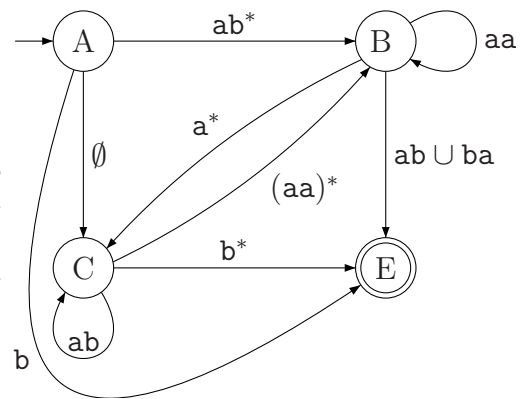
1 From NFA to regular expression

1.1 GNFA— A Generalized NFA

Consider an NFA N where we allowed to write any regular expression on the edges, and not only just symbols. The automata is allowed to travel on an edge, if it can matches a prefix of the unread input, to the regular expression written on the edge. We will refer to such an automata as a *GNFA* (*generalized non-deterministic finite automata* [Don't you just love all these shortcuts?]).

Thus, the GNFA on the right, accepts the string *abbbbaaba*, since

$$A \xrightarrow{abbbb} B \xrightarrow{aa} B \xrightarrow{ba} E.$$



To simplify the discussion, we would enforce the following conditions:

- C1 There are transitions going from the initial state to all other states, and there are no transitions into the initial state.
 1. There is a single accept state that has only transitions coming into it (and no outgoing transitions).
 2. The accept state is distinct from the initial state.
 3. Except for the initial and accepting states, all other states are connected to all other states via a transition. In particular, each state has a transition to itself.

When you can not actually go between two states, a GNFA has a transitions labelled with \emptyset , which will not match any string of input characters. We do not have to draw these transitions explicitly in the state diagrams.

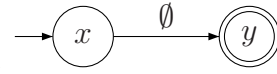
1.2 Top-level outline of conversion

We will convert a DFA to a regular expression as follows:

- (A) Convert DFA to a GNFA, adding new initial and final states.
- (B) Remove all states one-by-one, until we have only the initial and final states.
- (C) Output regex is the label on the (single) transition left in the GNFA. (The word *regex* is just a shortcut for regular expression.)

Lemma 1.1. *A DFA M can be converted into an equivalent GNFA G .*

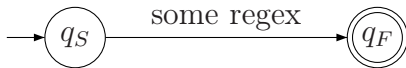
Proof: We can consider M to be an NFA. Next, we add a special initial state q_{init} that is connected to the old initial state via ε -transition. Next, we add a special final state q_{fin} , such that all the final states of M are connected to q_{fin} via an ε -transition. The modified NFA M' has an initial state and a single final state, such that no transition enters the initial state, and no transition leaves the final state, thus M' comply with conditions (C1–C3) above. Next, we consider all pair of states $x, y \in Q(M')$, and



if there is no transition between them, we introduce the transition. The resulting GNFA G from M' is now compliant also with condition (C4).

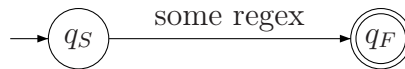
It is easy now to verify that G is equivalent to the original DFA M . ■

We will remove all the intermediate states from the GNFA, leaving a GNFA with only initial and final states, connected by one transition with a (typically complex) label on it. The equivalent regular expression is obvious: the label on the transition.



Lemma 1.2. *Given a GNFA N with $k = 2$ states, one can generate an equivalent regular expression.*

Proof: A GNFA with only two states (that comply with conditions (C1)-(C4)) have the following form.



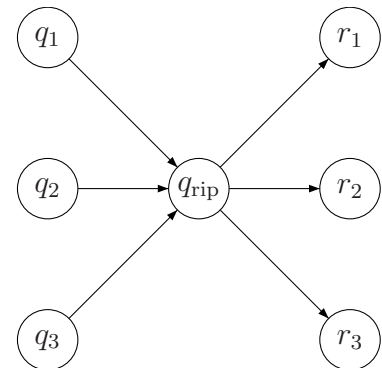
The GNFA has a single transition from the initial state to the accepting state, and this transition has the regular expression R associated with it. Since the initial state and the accepting state do not have self loops, we conclude that N accepts all words that matches the regular expression R . Namely, $L(N) = L(R)$. ■

1.3 Details of ripping out a state

We first describe the construction. Since $k > 2$, there is at least one state in N which is not initial or accepting, and let q_{rip} denote this state. We will “rip” this state out of N and fix the GNFA, so that we get a GNFA with one less state.

Transition paths going through q_{rip} might come from any of a variety of states q_1, q_2 , etc. They might go from q_{rip} to any of another set of states r_1, r_2 , etc.

For each pair of states q_i and r_i , we need to convert the transition through q_{rip} into a direct transition from q_i to r_i .



1.3.1 Reworking connections for specific triple of states

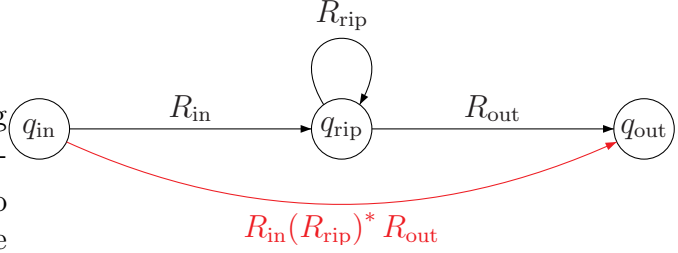
To understand how this works, let us focus on the connections between q_{rip} and two other specific states q_{in} and q_{out} . Notice that q_{in} and q_{out} might be the same state, but they both have to be different from q_{rip} .

The state q_{rip} has a self loop with regular expression R_{rip} associated with it. So, consider a fragment of an accepting trace that goes through q_{rip} . It transition into q_{rip} from a state q_{in} with a regular

expression R_{in} and travels out of q_{rip} into state q_{out} on an edge with the associated regular expression being R_{out} . This trace, corresponds to the regular expression R_{in} followed by 0 or more times of traveling on the self loop (R_{rip} is used each time we traverse the loop), and then a transition out to q_{out} using the regular expression R_{out} . As such, we can introduce a direct transition from q_{in} to q_{out} with the regular expression

$$R = R_{\text{in}}(R_{\text{rip}})^* R_{\text{out}}.$$

Clearly, any fragment of a trace traveling $q_{\text{in}} \rightarrow q_{\text{rip}} \rightarrow q_{\text{out}}$ can be replaced by the direct transition $q_{\text{in}} \xrightarrow{R} q_{\text{out}}$. So, let us do this replacement for any two such stages, we connect them directly via a new transition, so that they no longer need to travel through q_{rip} .



Clearly, if we do that for all such pairs, the new automata accepts the same language, but no longer need to use q_{rip} . As such, we can just remove q_{rip} from the resulting automata. And let M' denote the resulting automata.

The automata M' is not quite legal, yet. Indeed, we will have now parallel transitions because of the above process (we might even have parallel self loops). But this is easy to fix: We replace two such parallel transitions $q_i \xrightarrow{R_1} q_j$ and $q_i \xrightarrow{R_2} q_j$, by a single transition

$$q_i \xrightarrow{R_1 + R_2} q_j.$$

As such, for the triple $q_{\text{in}}, q_{\text{rip}}, q_{\text{out}}$, if the original label on the direct transition from q_{in} to q_{out} was originally R_{dir} , then the output label for the new transition (that skips q_{rip}) will be

$$R_{\text{dir}} + R_{\text{in}}(R_{\text{rip}})^* R_{\text{out}}. \quad (1)$$

Clearly the new transition, is equivalent to the two transitions it replaces. If we repeat this process for all the parallel transitions, we get a new GNFA M which has $k - 1$ states, and furthermore it accepts exactly the same language as N .

1.4 Proof of correctness of the ripping process

Lemma 1.3. *Given a GNFA N with $k > 2$ states, one can generate an equivalent GNFA M with $k - 1$ states.*

Proof: Since $k > 2$, N contains least one state in N which is not accepting, and let q_{rip} denote this state. We will “rip” this state out of N and fix the GNFA, so that we get a GNFA with one less state.

For every pair of states q_{in} and q_{out} , both distinct from q_{rip} , we replace the transitions that go through q_{rip} with direct transitions from q_{in} to q_{out} , as described in the previous section.

Correctness. Consider an accepting trace T for N for a word w . If T does not use the state q_{rip} than the same trace exactly is an accepting trace for M . So, assume that it uses q_{rip} , in particular, the trace looks like

$$T = \dots q_i \xrightarrow{S_i} q_{\text{rip}} \xrightarrow{S_{i+1}} q_{\text{rip}} \dots \xrightarrow{S_{j-1}} q_{\text{rip}} \xrightarrow{S_{j-1}} q_j \dots$$

0 or more times

Where $S_i S_{i+1} \dots, S_j$ is a substring of w . Clearly, $S_i \in R_{\text{in}}$, where R_{in} is the regular expression associated with the transition $q_i \rightarrow q_{\text{rip}}$. Similarly, $S_{j-1} \in R_{\text{out}}$, where R_{out} is the regular expression associated

with the transition $q_{rip} \rightarrow q_j$. Finally, $S_{i+1}S_{i+2}\cdots S_{j-1} \in (R_{rip})^*$, where R_{rip} is the regular expression associated with the self loop of q_{rip} .

Now, clearly, the string $S_iS_{i+1}\cdots S_j$ matches the regular expression $R_{in}(R_{out})^*R_{out}$. in particular, we can replace this portion of the trace of T by

$$T = \dots q_i \xrightarrow{S_iS_{i+1}\cdots S_{j-1}S_j} q_j \dots$$

This transition is using the new transition between q_i and q_j introduced by our construction. Repeating this replacement process in T till all the appearances of q_{rip} are removed, results in an accepting trace \hat{T} of M . Namely, we proved that any string accepted by N is also accepted by M .

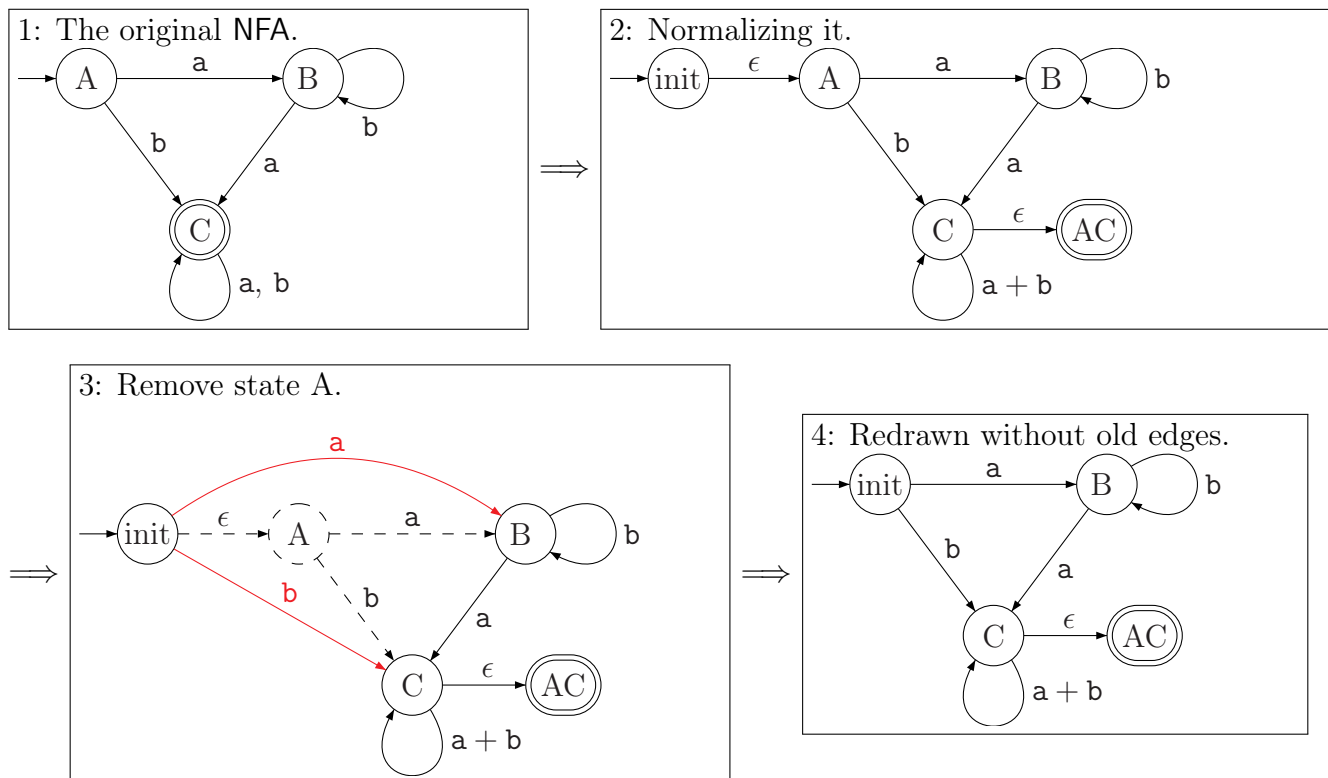
We need also to prove the other direction. Namely, given an accepting trace for M , we can rewrite it into an equivalent trace of N which is accepting. This is easy, and done in a similar way to what we did above. Indeed, if a portion of the trace uses a new transition of M (that does not appear in N), we can place it by a fragment of transitions going through q_{rip} . In light of the above proof, it is easy and we omit the straightforward but tedious details. ■

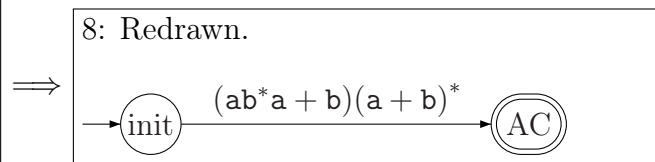
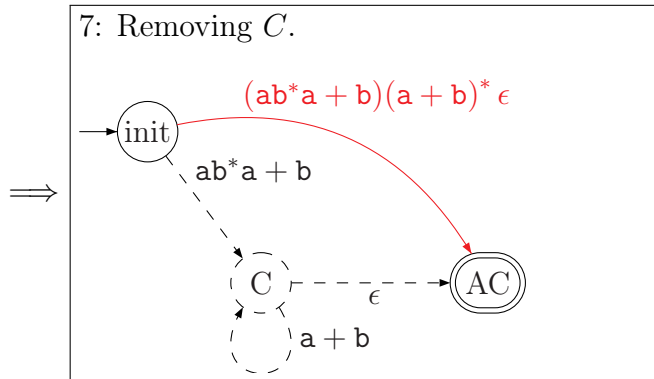
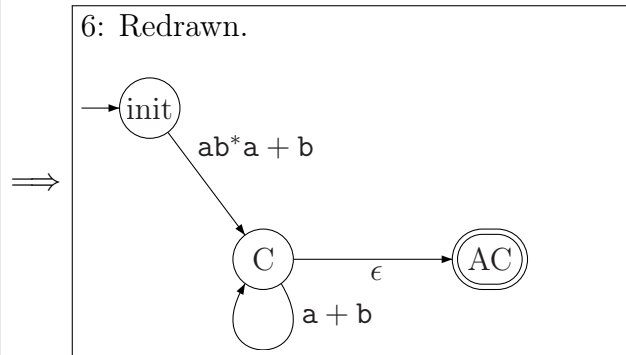
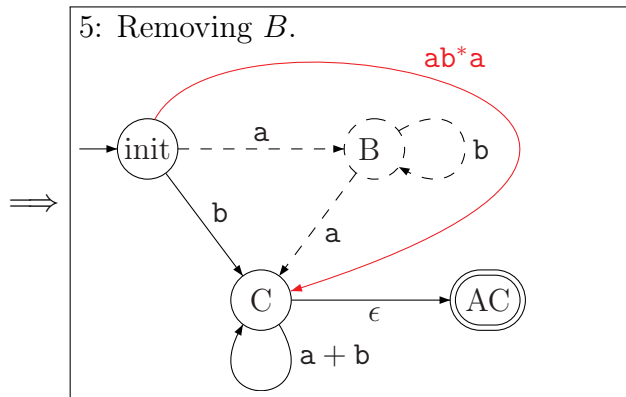
Theorem 1.4. *Any DFA can be translated into an equivalent regular expression.*

Proof: Indeed, convert the DFA into a GNFA N . As long as N has more than two states, reduce its number of states by removing one of its states using Lemma ???. Repeat this process till N has only two states. Now, we convert this GNFA into an equivalent regular expression using Lemma ???.

2 Examples

2.1 Example: From GNFA to regex in 8 easy figures





Thus, this automata is equivalent to the regular expression $(ab^*a + b)(a + b)^*$.