

Context Free Languages and Grammars

Lecture 7

Tuesday, February 5, 2019

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Regular Languages

DI

- Regular expressions allow us to describe/express a class of languages compactly and precisely.
- Equivalence with DFAs show the following: given any regular expression r there is a very efficient algorithm for solving the language recognition problem for $L(r)$: given $w \in \Sigma^*$ is $w \in L(r)$?

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Disadvantage of regular expressions/languages:

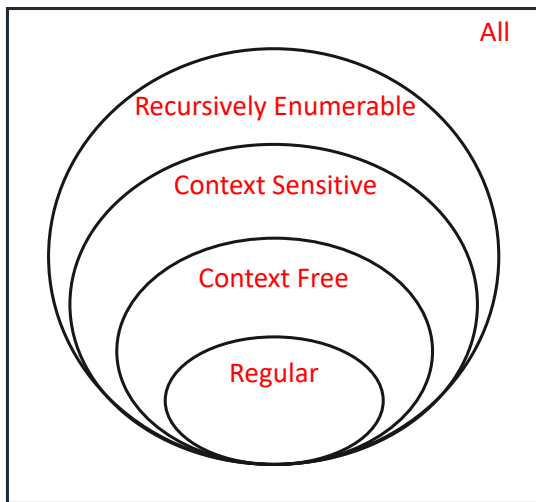
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Disadvantage of regular expressions/languages: too simple and cannot express interesting features such as balanced parenthesis that we need in programming languages. No recursion allowed even in limited form.

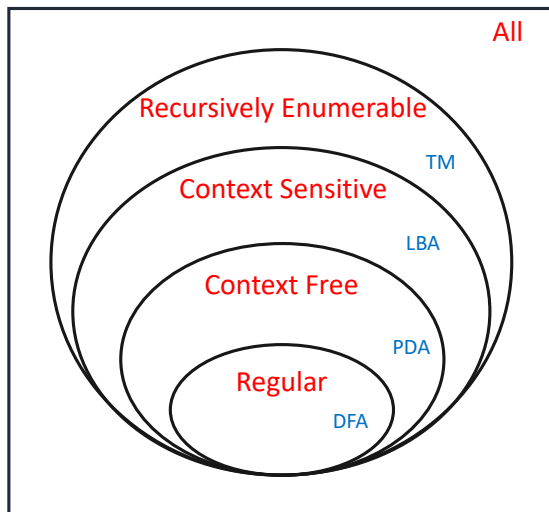
Language classes: Chomsky Hierarchy

Generative models for languages based on grammars.



Chomsky Hierarchy and Machines

For each class one can define a corresponding class of machines.



Regular vs. Context Free Languages

Regular Languages: Built from strings using:

- 1 Sequencing $A \cdot B$
- 2 Branching $A + B$
- 3 Repetition A^*

Regular vs. Context Free Languages

Regular Languages: Built from strings using:

- 1 Sequencing
- 2 Branching
- 3 Repetition

$$\omega = 0^*$$

$$\omega \begin{cases} \rightarrow \omega = \epsilon \\ \rightarrow \omega = 0\omega \end{cases}$$

Context Free Languages: Built from strings using:

- 1 Sequencing
- 2 Branching
- 3 Recursion

What stack got to do with it?

What's a stack but a second hand memory?

- ① **DFA/NFA**/Regular expressions.
≡ constant memory computation.
- ② Turing machines **DFA/NFA** + unbounded memory.
≡ a standard computer/program.

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≡ context free grammars (**CFG**).
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- ③ Turing machines **DFA/NFA** + unbounded memory.
≡ a standard computer/program.
≡ **NFA** with two stacks.

Programming Language Design

Question: What is a valid C program? Or a Python program?

Question: Given a string w what is an algorithm to check whether w is a valid C program? The parsing problem.

Context Free Languages and Grammars

- Programming Language Specification
- Parsing
- Natural language understanding
- Generative model giving structure
- ...

CFLs provide a good balance between expressivity and tractability.
Limited form of recursion.

Programming Languages

```
<relational-expression> ::= <shift-expression>
| <relational-expression> < <shift-expression>
| <relational-expression> > <shift-expression>
| <relational-expression> <= <shift-expression>
| <relational-expression> >= <shift-expression>

<shift-expression> ::= <additive-expression>
| <shift-expression> << <additive-expression>
| <shift-expression> >> <additive-expression>

<additive-expression> ::= <multiplicative-expression>
| <additive-expression> + <multiplicative-expression>
| <additive-expression> - <multiplicative-expression>

<multiplicative-expression> ::= <cast-expression>
| <multiplicative-expression> * <cast-expression>
| <multiplicative-expression> / <cast-expression>
| <multiplicative-expression> % <cast-expression>

<cast-expression> ::= <unary-expression>
| ( <type-name> ) <cast-expression>

<unary-expression> ::= <postfix-expression>
| ++ <unary-expression>
| -- <unary-expression>
| <unary-operator> <cast-expression>
| sizeof <unary-expression>
| sizeof <type-name>

<postfix-expression> ::= <primary-expression>
| <postfix-expression> [ <expression> ]
| <postfix-expression> ( {<assignment-expression>}* )
| <postfix-expression> . <identifier>
| <postfix-expression> -> <identifier>
| <postfix-expression> ++
| <postfix-expression> --
```

Natural Language Processing

English sentences can be described as

$$\begin{aligned}\langle S \rangle &\rightarrow \langle NP \rangle \langle VP \rangle \\ \langle NP \rangle &\rightarrow \langle CN \rangle \mid \langle CN \rangle \langle PP \rangle \\ \langle VP \rangle &\rightarrow \langle CV \rangle \mid \langle CV \rangle \langle PP \rangle \\ \langle PP \rangle &\rightarrow \langle P \rangle \langle CN \rangle \\ \langle CN \rangle &\rightarrow \langle A \rangle \langle N \rangle \\ \langle CV \rangle &\rightarrow \langle V \rangle \mid \langle V \rangle \langle NP \rangle \\ \langle A \rangle &\rightarrow a \mid the \\ \langle N \rangle &\rightarrow boy \mid girl \mid flower \\ \langle V \rangle &\rightarrow touches \mid likes \mid sees \\ \langle P \rangle &\rightarrow with\end{aligned}$$

English Sentences

Examples

$$\begin{array}{c} \text{noun-phrs} \quad \text{verb-phrs} \\ \underbrace{\quad a \quad boy} \quad \underbrace{\quad sees} \\ \text{article noun} \quad \text{verb} \end{array}$$

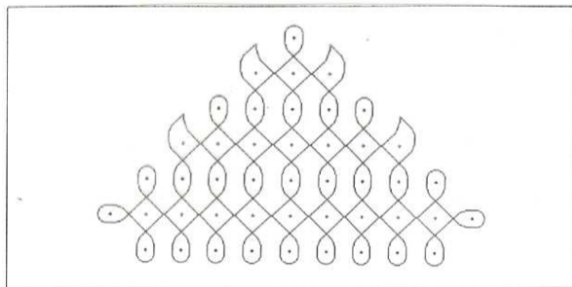
$$\begin{array}{c} \text{noun-phrs} \quad \text{verb-phrs} \\ \underbrace{\quad the \quad boy} \quad \underbrace{\quad sees \quad a \quad flower} \\ \text{article noun} \quad \text{verb} \quad \text{noun-phrs} \end{array}$$

Models of Growth

- *L*-systems
- <http://www.kevs3d.co.uk/dev/lsystems/>



Kolam drawing generated by grammar



Context Free Grammar (CFG) Definition

Definition

A **CFG** is a quadruple $G = (V, T, P, S)$

- V is a finite set of **non-terminal symbols**

$$G = \left(\text{Variables, Terminals, Productions, Start var} \right)$$

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- P is a finite set of **productions**, each of the form

$$A \rightarrow \alpha$$

where $A \in V$ and α is a string in $(V \cup T)^*$.

Formally, $P \subset V \times (V \cup T)^*$.

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- $S \in V$ is a **start symbol**

$$G = \left(\text{Variables, Terminals, Productions, Start var} \right)$$

Example

→ • $V = \{S\}$

→ • $T = \{a, b\}$ *OR*

→ • $P = \{S \rightarrow \epsilon \mid a \mid b \mid aSa \mid bSb\}$

(abbrev. for $S \rightarrow \epsilon, S \rightarrow a, S \rightarrow b, S \rightarrow aSa, S \rightarrow bSb$)

Example

- $V = \{S\}$
- $T = \{a, b\}$
- $P = \{S \rightarrow \epsilon \mid a \mid b \mid aSa \mid bSb\}$
(abbrev. for $S \rightarrow \epsilon, S \rightarrow a, S \rightarrow b, S \rightarrow aSa, S \rightarrow bSb$)

$S \rightsquigarrow aSa \rightsquigarrow abSba \rightsquigarrow abbSbba \rightsquigarrow abb\ b\ bba$
 $abba\ bba$

Example

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 $S \rightarrow bS$

$S \rightsquigarrow aSa \rightsquigarrow abSba \rightsquigarrow abbSbba \rightsquigarrow \underline{abb b bba}$

What strings can S generate like this?

Palindromes

$\{a, b\}$

$$w = w^R$$

Example formally...

- $V = \{S\}$
- $T = \{a, b\}$
- $P = \{S \rightarrow \epsilon \mid a \mid b \mid aSa \mid bSb\}$
(abbrev. for $S \rightarrow \epsilon, S \rightarrow a, S \rightarrow b, S \rightarrow aSa, S \rightarrow bSb$)

$$G = \left(\{S\}, \{a, b\}, \left\{ \begin{array}{l} S \rightarrow \epsilon, \\ S \rightarrow a, \\ S \rightarrow b \\ S \rightarrow aSa \\ S \rightarrow bSb \end{array} \right\}, S \right)$$

Palindromes

- Madam in Eden I'm Adam
- Dog doo? Good God!
- Dogma: I am God.
- A man, a plan, a canal, Panama
- Are we not drawn onward, we few, drawn onward to new era?
- Doc, note: I dissent. A fast never prevents a fatness. I diet on cod.
- <http://www.palindromelist.net>

Examples

$$L = \{0^n 1^n \mid n \geq 0\}$$

$$G = \{V, T, P, S\}$$

$$V = \{S\}$$

$$T = \{0, 1\}$$

$$P = \{S \rightarrow \epsilon \mid 0S1\}$$

$$S \rightarrow \epsilon$$

$$S \rightarrow 0S1 \rightarrow 01$$

$$S \rightarrow 0S1 \rightarrow 00S11 \xrightarrow{S \rightarrow \epsilon} 0011$$

$$S \rightarrow 0^n 1^n$$

Examples

$$L = \{0^n 1^n \mid n \geq 0\}$$

$$S \rightarrow \epsilon \mid \mathbf{0S1}$$


Notation and Convention

Let $G = (V, T, P, S)$ then

- a, b, c, d, \dots , in T (terminals)
- A, B, C, D, \dots , in V (non-terminals)
- u, v, w, x, y, \dots in \underline{T}^* for strings of terminals
- $\alpha, \beta, \gamma, \dots$ in $(V \cup T)^*$
- X, Y, X in $V \cup T$

“Derives” relation

Formalism for how strings are derived/generated

Definition

Let $G = (V, T, P, S)$ be a CFG. For strings $\alpha_1, \alpha_2 \in (V \cup T)^*$ we say α_1 **derives** α_2 denoted by $\alpha_1 \rightsquigarrow_G \alpha_2$ if there exist strings β, γ, δ in $(V \cup T)^*$ such that

- $\alpha_1 = \beta \underline{A} \delta$
- $\alpha_2 = \beta \gamma \delta$
- $A \rightarrow \gamma$ is in P .

Examples: $S \rightsquigarrow \epsilon$, $S \rightsquigarrow 0S1$, $0S1 \rightsquigarrow 00S11$, $0S1 \rightsquigarrow 01$.

$$\alpha_1 = 0S1$$

$$\alpha_2 = 00\overline{S}11$$

$$\alpha_1 \rightsquigarrow \alpha_2$$

$$S \rightarrow 0S1$$

“Derives” relation continued

Definition

For integer $k \geq 0$, $\alpha_1 \rightsquigarrow^k \alpha_2$ inductive defined:

- $\alpha_1 \rightsquigarrow^0 \alpha_2$ if $\alpha_1 = \alpha_2$
- $\alpha_1 \rightsquigarrow^k \alpha_2$ if $\alpha_1 \rightsquigarrow \beta_1$ and $\beta_1 \rightsquigarrow^{k-1} \alpha_2$.

“Derives” relation continued

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- **Alternative definition:** $\alpha_1 \rightsquigarrow^k \alpha_2$ if $\alpha_1 \rightsquigarrow^{k-1} \beta_1$ and $\beta_1 \rightsquigarrow \alpha_2$

S \rightsquigarrow 008111

“Derives” relation continued

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\rightsquigarrow^* is the reflexive and transitive closure of \rightsquigarrow .

$\alpha_1 \rightsquigarrow^* \alpha_2$ if $\alpha_1 \rightsquigarrow^k \alpha_2$ for some k .

Examples: $S \rightsquigarrow^* \epsilon$, $0S1 \rightsquigarrow^* 0000011111$.

Context Free Languages

Definition

The language generated by CFG $G = (V, T, P, S)$ is denoted by $L(G)$ where $L(G) = \{w \in T^* \mid S \rightsquigarrow^* w\}$.

Context Free Languages

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Definition

A language L is **context free** (CFL) if it is generated by a context free grammar. That is, there is a CFG G such that $L = L(G)$.

Example

$$L = \{0^n 1^n \mid n \geq 0\} \quad P = \{S \rightarrow \epsilon \mid 0S1\}$$

$$L(G) = L$$

$$L(G) \subseteq L, \quad L \subseteq L(G)$$

* $\forall n, w = 0^n 1^n$, there is $S \xrightarrow{*} w$

$n=0$; $w = 0^0 1^0 = \epsilon$; $S \rightarrow \epsilon$

I.H. for $k < n$; $S \xrightarrow{*} 0^k 1^k$

$$S \xrightarrow{*} 0S1 \xrightarrow{*} 00^k 1^k 1$$

$$G = \{ \{S\}, \{0, 1\}, P, S \}$$

$$S \rightarrow \epsilon \mid SS$$

$$S \rightarrow SS \rightarrow S \rightarrow SS$$

$$S \rightarrow 0A11\epsilon \quad \left. \begin{array}{l} L(S) \\ L(A) \end{array} \right\}$$

$$A \rightarrow 0S11\epsilon$$

* $\forall w \in L(G)$; $w = 0^n 1^n$ for some n

$$S \xrightarrow{*} w \Rightarrow w = \epsilon = 0^0 1^0$$

let w be an arbitrary string $\in L(G) \Rightarrow S \xrightarrow{*} w$

I.H. for any $k < n \Rightarrow S \xrightarrow{*} u$; $w = 0^k 1^k$

$$S \xrightarrow{*} w \Rightarrow S \xrightarrow{*} 0S1 \xrightarrow{*} 0u1$$

$$u = 0^{k-1} 1^{k-1}$$

Example

$$L = \{0^n 1^n \mid n \geq 0\} \quad S_1 \rightarrow \epsilon \mid 0S_1 \mid 1S_1$$

$$L = 0^* 1^* \quad S \rightarrow \epsilon \mid 0S \mid 1S$$

Example

$$L = \{0^n 1^n \mid n \geq 0\} \quad S_1 \rightarrow \epsilon \mid 0S_11$$

$$L = 0^*1^* \quad S_2 \rightarrow \epsilon \mid 0S_2 \mid S_21$$

$$L = \{0^n 1^m \mid m > n\} = 0^{\wedge} 1^{\wedge} 1^+$$

$$\begin{aligned} \epsilon &= 0^0 1^0 \\ 0 &= 0 \\ n &= m \end{aligned}$$

$$S \rightarrow 1 \mid S1 \mid 0S1$$

$$S \rightarrow A1$$

$$A \rightarrow \epsilon \mid 0A1 \mid A1$$

$$S \rightarrow S1 \mid A1$$

$$A \rightarrow \epsilon \mid 0A1$$

Example

$$L = \{0^n 1^n \mid n \geq 0\}$$

$$L = 0^* 1^*$$

$$L = \{0^n 1^m \mid m > n\} \quad S \rightarrow S \mid A \quad , A \rightarrow \epsilon \mid 0 A \underline{1}$$

$$L = \{0^n 1^m \mid m < n\} \quad S \rightarrow 0 S \mid 0 A \quad , A \rightarrow \epsilon \mid 0 A \underline{1}$$

Example

$$L = \{0^n 1^n \mid n \geq 0\}$$

$$L = 0^* 1^*$$

$$L = \{0^n 1^m \mid m > n\} \quad B \rightarrow B1 \mid A \ ; \ A \rightarrow \epsilon \mid 0A \mid$$

$$L = \{0^n 1^m \mid m < n\} \quad C \rightarrow 0C \mid 0A \ ; \ A \rightarrow \epsilon \mid 0A \mid$$

$$L = \{0^n 1^m \mid m \neq n\} \quad P = \left. \begin{array}{l} S \rightarrow B1C \\ B \rightarrow B1 \mid A \\ C \rightarrow 0C \mid 0A \\ A \rightarrow \epsilon \mid 0A \end{array} \right\} \quad V = \{S, A, B, C\}$$
$$T = \{0, 1\}$$

Example

$L = \{w \in \{(\underline{\quad}, \underline{\quad})\}^* \mid w \text{ is properly nested string of parenthesis}\}$

~~$S \rightarrow \epsilon \mid (S) \mid ()S \mid S()$~~ can derive
 $((\quad)(\quad))$

$S \rightarrow \epsilon \mid (S) \mid SS$

$S \rightarrow (S)$

$\rightarrow ((S))$

$\rightarrow ((\quad)(\quad))$

$\rightarrow ((\quad)(\quad))$

$S \rightarrow \epsilon \mid (S) \mid (S)S$

cannot derive
 $((\quad))(\quad)$

Example

$$L = \{w \in \{(,)\}^* \mid w \text{ is properly nested string of parenthesis}\}$$

$$L = \{w \in \{0,1\}^* \mid w \text{ has equal number of 1s as 0's}\}$$

$$S \rightarrow \epsilon \mid SOSI \mid SISO$$

$$S \rightarrow \epsilon \mid OSI \mid ISO \mid SS$$

$$\rightarrow S \rightarrow \epsilon \mid \cancel{OSI} \mid \cancel{ISO} \mid \cancel{SIS} \mid OSI \mid SIO \mid SOI$$

$\rightarrow \underline{||000000||}$

$$S \rightarrow OSI, ISO$$

$$S \rightarrow IOS, OIS, SIO, SOI$$

Closure Properties of CFLs

$G_1 = (V_1, T, P_1, S_1)$ and $G_2 = (V_2, T, P_2, S_2)$

Assumption: $V_1 \cap V_2 = \emptyset$, that is, non-terminals are not shared

Closure Properties of CFLs

$G_1 = (V_1, T, P_1, S_1)$ and $G_2 = (V_2, T, P_2, S_2)$

Assumption: $V_1 \cap V_2 = \emptyset$, that is, non-terminals are not shared

Theorem

CFLs are closed under union. L_1, L_2 CFLs implies $L_1 \cup L_2$ is a CFL.

Theorem

CFLs are closed under concatenation. L_1, L_2 CFLs implies $L_1 \cdot L_2$ is a CFL.

Theorem

CFLs are closed under Kleene star.
If L is a CFL $\implies L^*$ is a CFL.

Closure Properties of CFLs

Union

$$G_1 = (V_1, T, P_1, S_1) \text{ and } G_2 = (V_2, T, P_2, S_2)$$

Assumption: $V_1 \cap V_2 = \emptyset$, that is, non-terminals are not shared.

Theorem

CFLs are closed under union. L_1, L_2 **CFLs** implies $L_1 \cup L_2$ is a **CFL**.

$$G = \{V, T, P, S\}$$

$$V = V_1 \cup V_2 \cup \{S\}$$

$$T = T$$

$$P = P_1 \cup P_2 \cup \{S \rightarrow S_1 \mid S_2\}$$

$$S = S$$

Closure Properties of CFLs

Concatenation

Theorem

CFLs are closed under concatenation. L_1, L_2 CFLs implies $L_1 \bullet L_2$ is a **CFL**.

$$G = \{V, T, P, S\}$$

$$V = V_1 \cup V_2 \cup \{S\}$$

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$$P = P_1 \cup P_2 \cup \{S \rightarrow S_1 S_2\}$$

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Closure Properties of CFLs

Stardom (i.e, Kleene star)

Theorem

CFLs are closed under Kleene star.

If L_1 is a **CFL** $\implies L_1^*$ is a **CFL**.

$$G = \{V, T, P, S\}$$

$$V = V_1 \cup \{\$\}$$

$$T = T$$

$$P = \{S \rightarrow \epsilon \mid S S_1\} \cup P_1$$

$$S = S$$

Exercise

- Prove that every regular language is context-free using previous closure properties.
- Prove the set of regular expressions over an alphabet Σ forms a non-regular language which is context-free.

$$\Sigma \cup \{+, \cdot, *, (,), \varepsilon, \emptyset\}$$

$$\emptyset \neq + \mid) \neq$$

$$\{(,)\} \cup \{+, \cdot, *, (,), \varepsilon, \emptyset\}$$

Closure Properties of CFLs continued

Theorem

CFLs are *not* closed under complement or intersection.

Theorem

If L_1 is a CFL and L_2 is regular then $L_1 \cap L_2$ is a CFL.

Theorem

$L = \{a^n b^n c^n \mid n \geq 0\}$ is not context-free.

Proof based on **pumping lemma** for CFLs. Technical and outside the scope of this class.

Parse Trees or Derivation Trees

A tree to represent the derivation $S \rightsquigarrow^* w$.

- Rooted tree with root labeled S
- Non-terminals at each internal node of tree
- Terminals at leaves
- Children of internal node indicate how non-terminal was expanded using a production rule

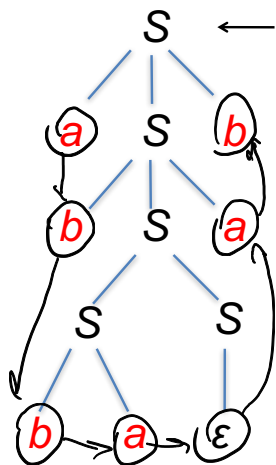
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- Children of internal node indicate how non-terminal was expanded using a production rule

A picture is worth a thousand words

Example



← A derivation tree for $abbaab$
(also called “parse tree”)

$S \rightarrow aSb \mid bSa \mid SS \mid ab \mid ba \mid \epsilon$

A corresponding derivation of $abbaab$



$S \rightarrow aSb \rightarrow abSab \rightarrow abSSab \rightarrow abbaSab \rightarrow abbaab$

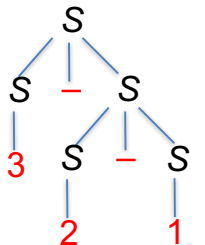
Ambiguity in CFLs

Definition

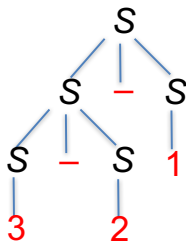
A CFG G is **ambiguous** if there is a string $w \in L(G)$ with two different parse trees. If there is no such string then G is **unambiguous**.

Example: $S \rightarrow S - S \mid 1 \mid 2 \mid 3$

$3 - 2 - 1$



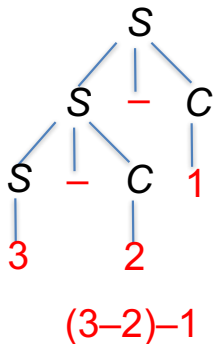
$$3 - (2 - 1) = 2$$



$$(3 - 2) - 1 = 0$$

Ambiguity in CFLs

- Original grammar: $S \rightarrow S - S \mid 1 \mid 2 \mid 3$
- Unambiguous grammar:
 $S \rightarrow S - C \mid 1 \mid 2 \mid 3$
 $C \rightarrow 1 \mid 2 \mid 3$



The grammar forces a parse corresponding to left-to-right evaluation.

Inherently ambiguous languages

Definition

A CFL L is **inherently ambiguous** if there is no unambiguous CFG G such that $L = L(G)$.

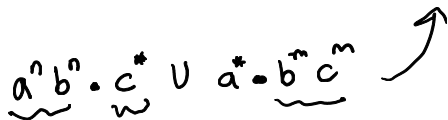
Inherently ambiguous languages

Definition

A CFL L is **inherently ambiguous** if there is no unambiguous CFG G such that $L = L(G)$.

- There exist inherently ambiguous CFLs.

Example: $L = \{a^n b^m c^k \mid n = m \text{ or } m = k\}$

$$\underbrace{a^n b^n} \cdot \underbrace{c^*} \cup a^* \cdot \underbrace{b^m c^m}$$


Inherently ambiguous languages

Definition

A CFL L is **inherently ambiguous** if there is no unambiguous CFG G such that $L = L(G)$.

- There exist inherently ambiguous CFLs.

Example: $L = \{a^n b^m c^k \mid n = m \text{ or } m = k\}$

- Given a grammar G it is **undecidable** to check whether $L(G)$ is inherently ambiguous. No algorithm!

Inductive proofs for CFGs

Question: How do we formally prove that a CFG $L(G) = L$?

Example: $S \rightarrow \epsilon \mid a \mid b \mid aSa \mid bSb$

Theorem

$$L(G) = \{\text{palindromes}\} = \{w \mid w = w^R\}$$

Inductive proofs for CFGs

Question: How do we formally prove that a CFG $L(G) = L$?

Example: $S \rightarrow \epsilon \mid a \mid b \mid aSa \mid bSb$

Theorem

$$L(G) = \{\text{palindromes}\} = \{w \mid w = w^R\}$$

Two directions:

- $L(G) \subseteq L$, that is, $S \rightsquigarrow^* w$ then $w = w^R$
- $L \subseteq L(G)$, that is, $w = w^R$ then $S \rightsquigarrow^* w$

$L(G) \subseteq L$

Show that if $S \rightsquigarrow^* w$ then $w = w^R$

By induction on **length of derivation**, meaning

For all $k \geq 1$, $S \rightsquigarrow^{*k} w$ implies $w = w^R$.

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By induction on **length of derivation**, meaning

For all $k \geq 1$, $S \rightsquigarrow^{*k} w$ implies $w = w^R$.

- If $S \rightsquigarrow^1 w$ then $w = \epsilon$ or $w = a$ or $w = b$. Each case $w = w^R$.
- Assume that for all $k < n$, that if $S \rightarrow^k w$ then $w = w^R$
- Let $S \rightsquigarrow^n w$ (with $n > 1$). Wlog w begin with a .
 - Then $S \rightarrow aSa \rightsquigarrow^{k-1} aua$ where $w = aua$.
 - And $S \rightsquigarrow^{n-1} u$ and hence IH, $u = u^R$.
 - Therefore $w^r = (aua)^R = (ua)^R a = au^R a = aua = w$.

$L \subseteq L(G)$

Show that if $w = w^R$ then $S \rightsquigarrow^* w$.

By induction on $|w|$

That is, for all $k \geq 0$, $|w| = k$ and $w = w^R$ implies $S \rightsquigarrow^* w$.

Exercise: Fill in proof.

Mutual Induction

Situation is more complicated with grammars that have multiple non-terminals.

See Section 5.3.2 of the notes for an example proof.

$$\left. \begin{array}{l} S \rightarrow \epsilon \mid OA \\ A \rightarrow \epsilon \mid IS \end{array} \right\} \text{mutual} \quad \begin{array}{l} L(S) = \\ L(A) = \end{array}$$

Normal Forms

Normal forms are a way to restrict form of production rules

Advantage: Simpler/more convenient algorithms and proofs

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Two standard normal forms for **CFGs**

- Chomsky normal form
- Greibach normal form

Normal Forms

Chomsky Normal Form:

- Productions are all of the form $A \rightarrow BC$ or $A \rightarrow a$.
If $\epsilon \in L$ then $S \rightarrow \epsilon$ is also allowed.
- Every CFG G can be converted into CNF form via an efficient algorithm
- Advantage: parse tree of constant degree.



Normal Forms

Chomsky Normal Form:

- Productions are all of the form $A \rightarrow BC$ or $A \rightarrow a$.
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- Every CFG G can be converted into CNF form via an efficient algorithm
- Advantage: parse tree of constant degree.

Greibach Normal Form:

- Only productions of the form $A \rightarrow a\beta$ are allowed.
- All CFLs without ϵ have a grammar in GNF. Efficient algorithm.
- Advantage: Every derivation adds exactly one terminal.

Language recognition for CFLs

Algorithmic question: Given CFG G and string $w \in \Sigma^*$ is $w \in L(G)$?

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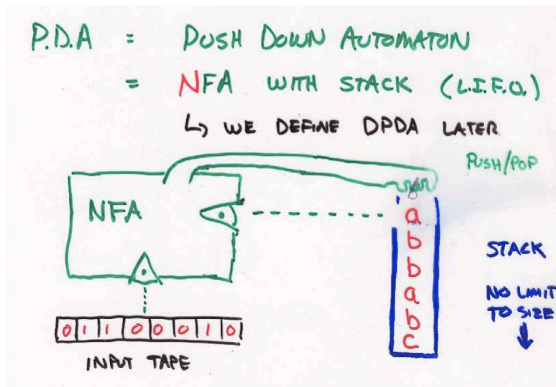
Later in course: algorithm for above problem that runs in $O(|w|^3)$ time for any fixed grammar G . Via dynamic programming.

Hence parsing problem for programming languages is solvable. However cubic time algorithm is too slow! For this reason grammars for PLs are restricted even further to make parsing algorithm faster (essentially linear time) — see CS 421 and compiler courses.

In programming languages some amount of “context” may be necessary. But CSL recognition is undecidable (no algorithm)! Hence people use ad hoc methods for the limited needs in PLs.

Things to know: Pushdown Automata

PDA: a NFA coupled with a stack



PDA's and CFG's are equivalent: both generate exactly CFL's.
PDA is a machine-centric view of CFL's.

Chomsky Hierarchy

See Wikipedia article for more on Chomsky Hierarchy including the grammar rules for Context Sensitive Languages etc.

https://en.wikipedia.org/wiki/Chomsky_hierarchy