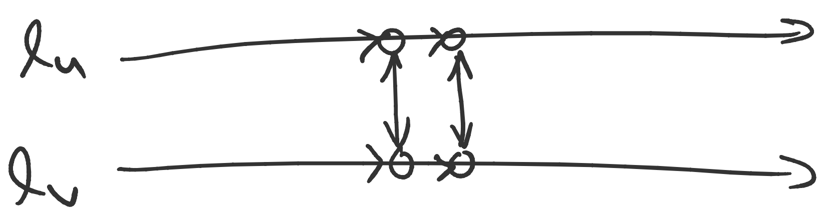
Given input ... integer k,

Construct input to dir-HC: dir graph G'.
as follows:

for each vertex $v \in V$,
draw a "line" l_v

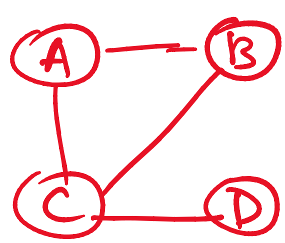
for each edge $uv \in E$,
add a gadget between l_u & l_v



create k extra vertices $z_1, \ldots, z_k$
going into 1st vertex of each line
& out of last vertex of each line

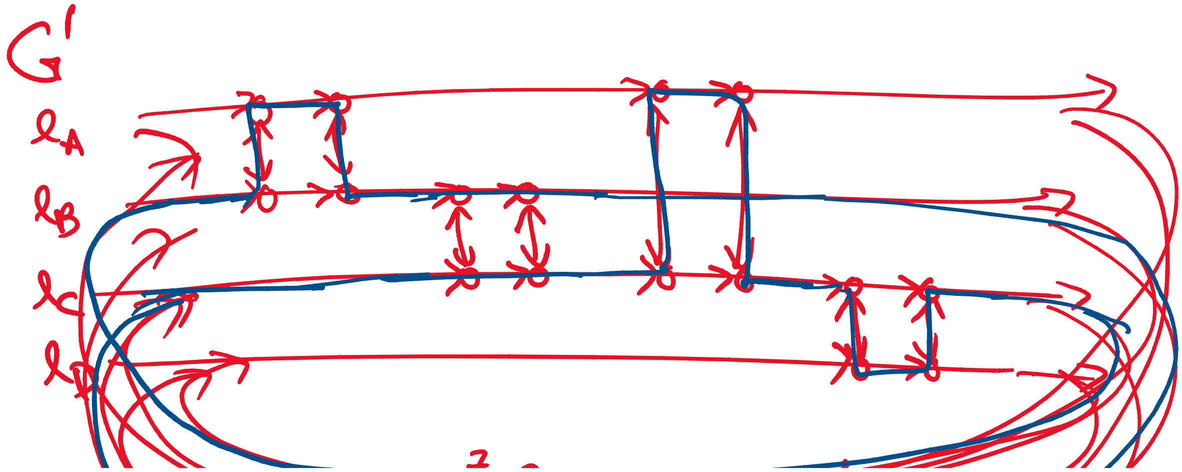
Construction from $(G, k) \to G'$ takes poly time.

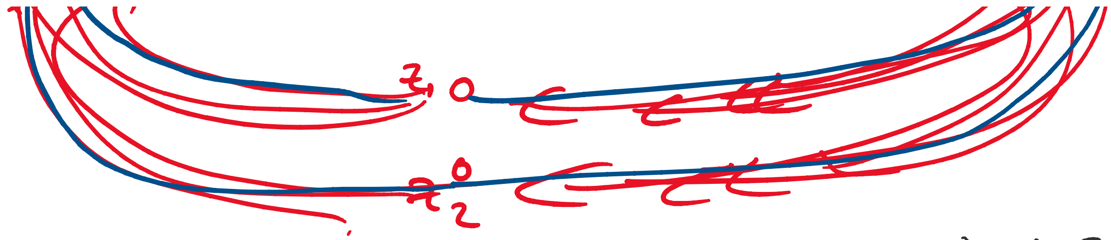
e.g. G



$k = 2$

$\{B, C\}$

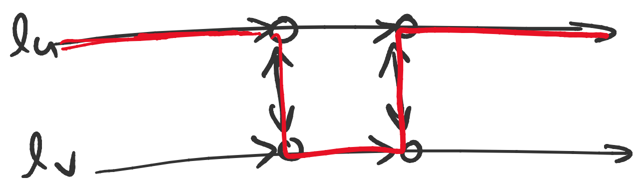




Correctness: $\exists$ vertex cover S of size $(\leq)k$ in G
$\iff$ $\exists$ Ham cycle C in G'

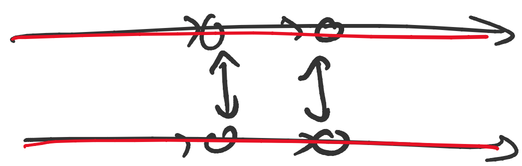
Pf: ($\Rightarrow$) Given vertex cover S of size k,
form k paths following l_u for each $u \in S$
inside gadget between l_u and l_v,

if $u \in S$, $v \notin S$,



if $u \notin S$, $v \in S$ — symmetric

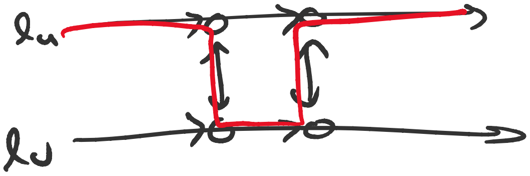
if $u \in S$, $v \in S$,



put these paths together using $z_1, \ldots, z_k$
to get a Ham cycle in G'.

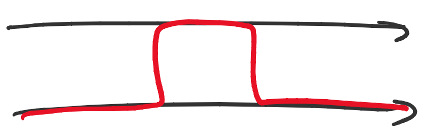
($\Leftarrow$) Given Ham cycle in G',

if

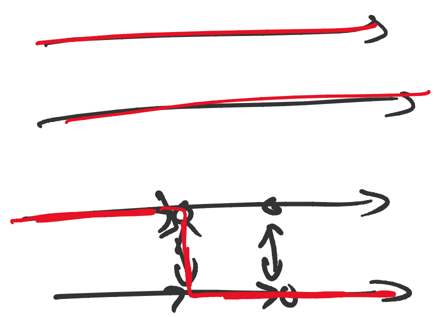


put $u \in S$, $v \notin S$

if



symmetric

if  put $u \in S$
$v \in S$

no other way ... $\square$

Subset Sum

Input: numbers $a_1, \dots, a_n, W$ $(0 < a_i \le W)$

Output: yes iff $\exists S \subseteq \{a_1, \dots, a_n\}$ that sums to W

Note - brute force $O(2^n)$ $(2^{n/2})$

DP $O(nW)$ time (input ~ $n \log W$)

not really polynomial

① Subset-Sum $\in$ NP:

Certificate: subset S ← poly size (n bits)

Certifier: check that sum of S is W

takes $O(n \log W)$ time
truly poly time

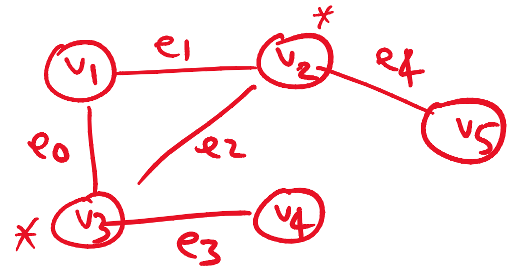
② Vertex-Cover $\le_p$ Subset-Sum:

Given input to Vertex-Cover: graph $G=(V,E)$ and integer k,

Construct input to Subset-Sum: set of numbers and W.

as follows:

e.g. G:



$k=2$

		e_4	e_3	e_2	e_1	e_0	
v_1	1	0	0	0	1	1	$= a_1$
* v_2	1	1	0	1	1	0	$= a_2$ *
* v_3	1	0	1	1	0	1	$= a_3$ *
v_4	1	0	1	0	0	0	$= a_4$
v_5	1	1	0	0	0	0	$= a_5$
						1	$= b_0$ *
					1	0	$= b_1$ *
				1	0	0	$= b_2$
			1	0	0	0	$= b_3$ *
		1	0	0	0	0	$= b_4$ *
	K	2	2	2	2	2	$= W$

Write $V = \{v_1, \ldots, v_n\}$, $E = \{e_0, \ldots, e_{m-1}\}$,

Let $c_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is incid to } v_i \\ 0 & \text{else} \end{cases}$

Let $a_i = 10^m + \sum_{j=0}^{m-1} c_{ij} 10^j$

$b_j = 10^j$

$W = K \cdot 10^m + \sum_{j=0}^{m-1} 2 \cdot 10^j$

This construction $(G, k) \to (a_1, \ldots, a_n, b_0, \ldots, b_{m-1}, W)$ takes polytime (each number has $O(m)$ digits)

Correctness: $\exists$ vertex cover S in G of size $(\leq)$ k
$\iff \exists$ subset $S' \subseteq \{a_1, \ldots, a_n, b_0, \ldots, b_{m-1}\}$

Correctness: ...

$\iff \exists$ subset $S' \subseteq \{a_1, \ldots, a_n, b_0, \ldots, b_{m-1}\}$ that sums to W.

Pf: ($\Rightarrow$) given vertex cover S,

define $S' = \{a_i : v_i \in S\} \cup \{b_j : e_j$ incident to exactly one vertex of $S\}$

Then sum of S' is W...

($\Leftarrow$) given S',

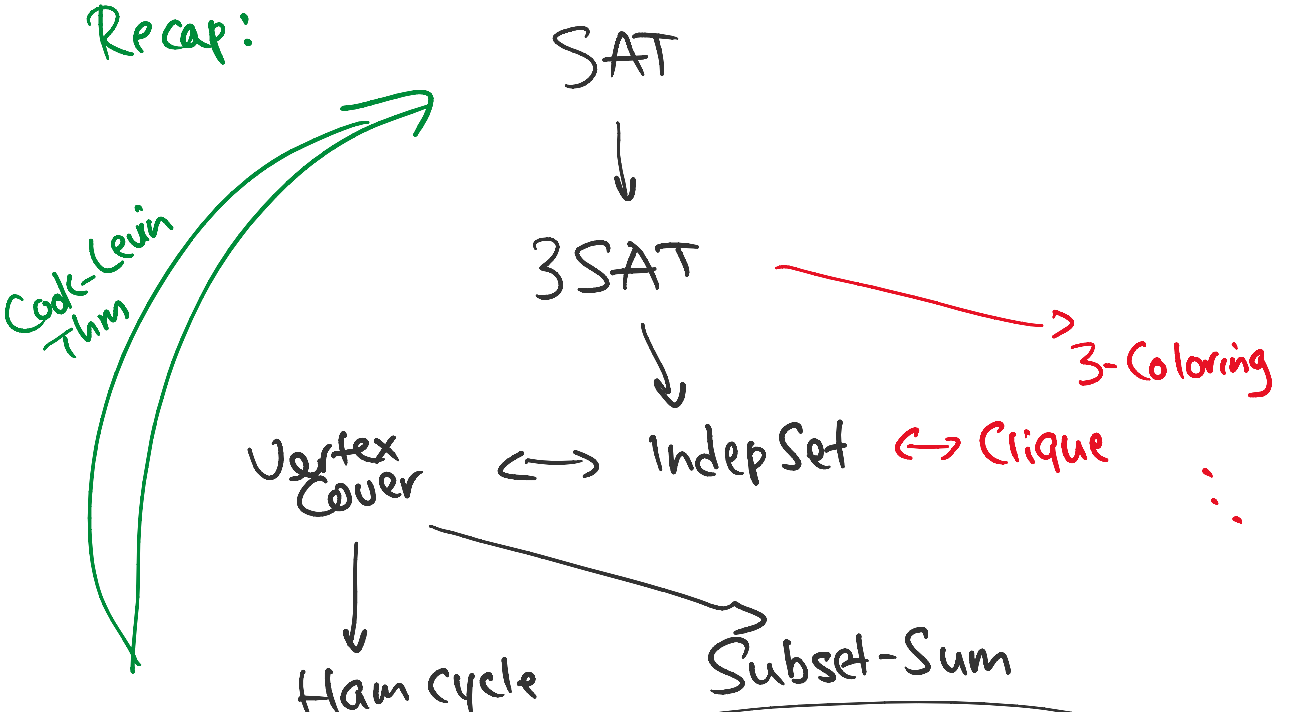
define $S = \{v_i : a_i \in S'\}$.

Then $|S| = k$ because of the leftmost column

& $\forall j$, e_j is incid to 1 or 2 vertices of S because of j^{th} column.

$\square$

Recap:

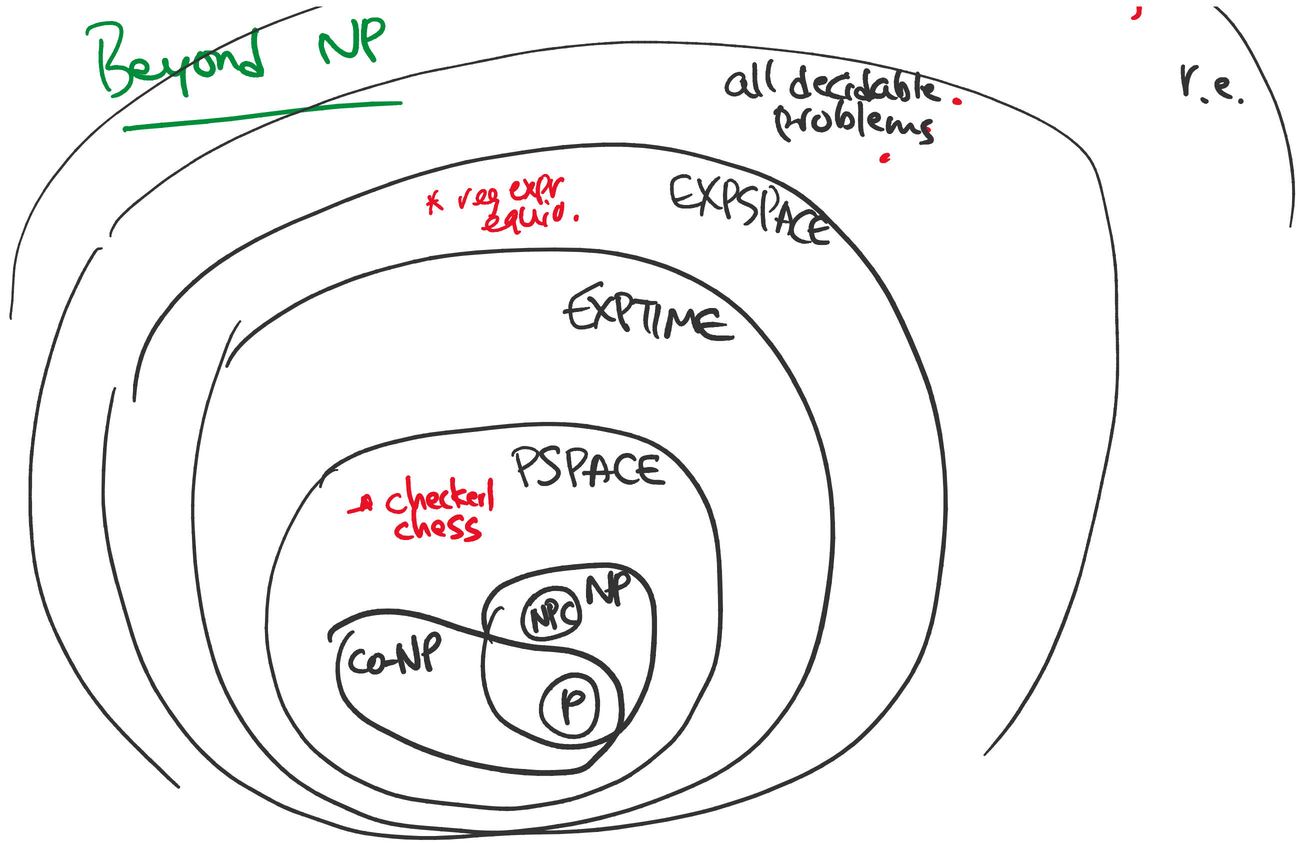


* Halting

Beyond NP

all decid...

Beyond NP



$P \neq EXPTIME$

$NPSPACE = PSPACE$ ..