April 4, 2019 10:59 AM

Midterm 2

Mon 7pm – 9pm

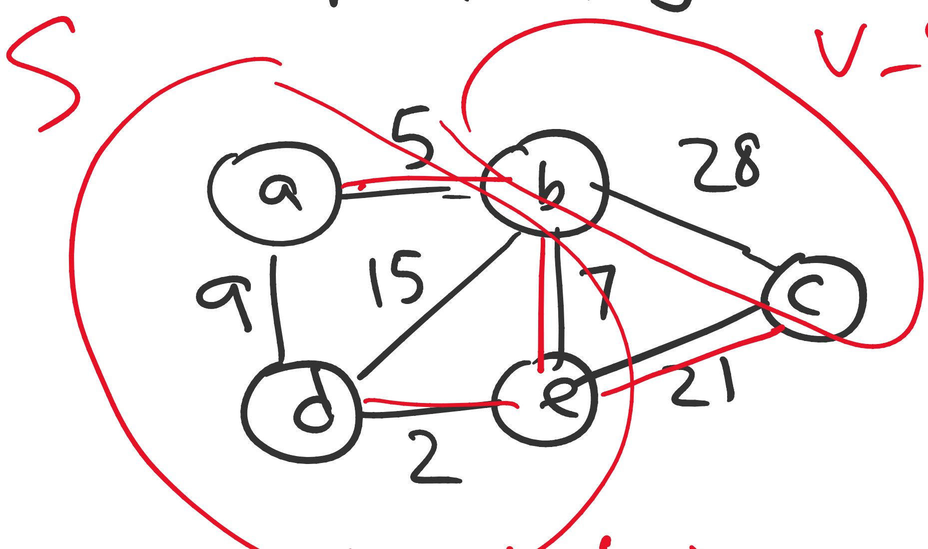
Covers everything after mid 1 to last wk

cheat sheet

Min Spanning Tree (MST)

Given weighted undir. connected graph $G=(V,E)$,
$w: E \to \mathbb{R}^{+}$,

find a connected subgraph that
includes all vertices
minimizing total wt



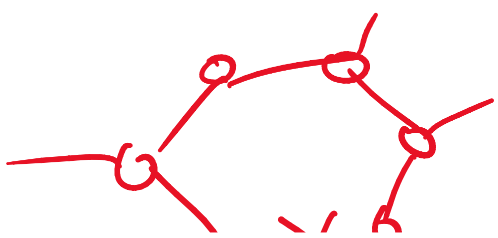
appl. network design

one sol'n:
$5+9+7+28$
$= 49$

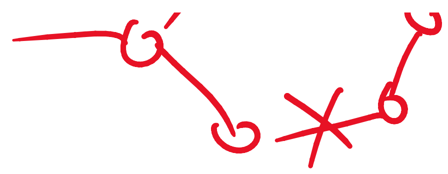
better sol'n:
$5+7+2+21$
$= 35$
optimal

Obs The opt subgraph must be acyclic.

Pf sketch:



$\equiv$ undir. acyclic connected



undir. acyclic connected
$\equiv$
tree

idea 0 - brute force $\Rightarrow$ exponential

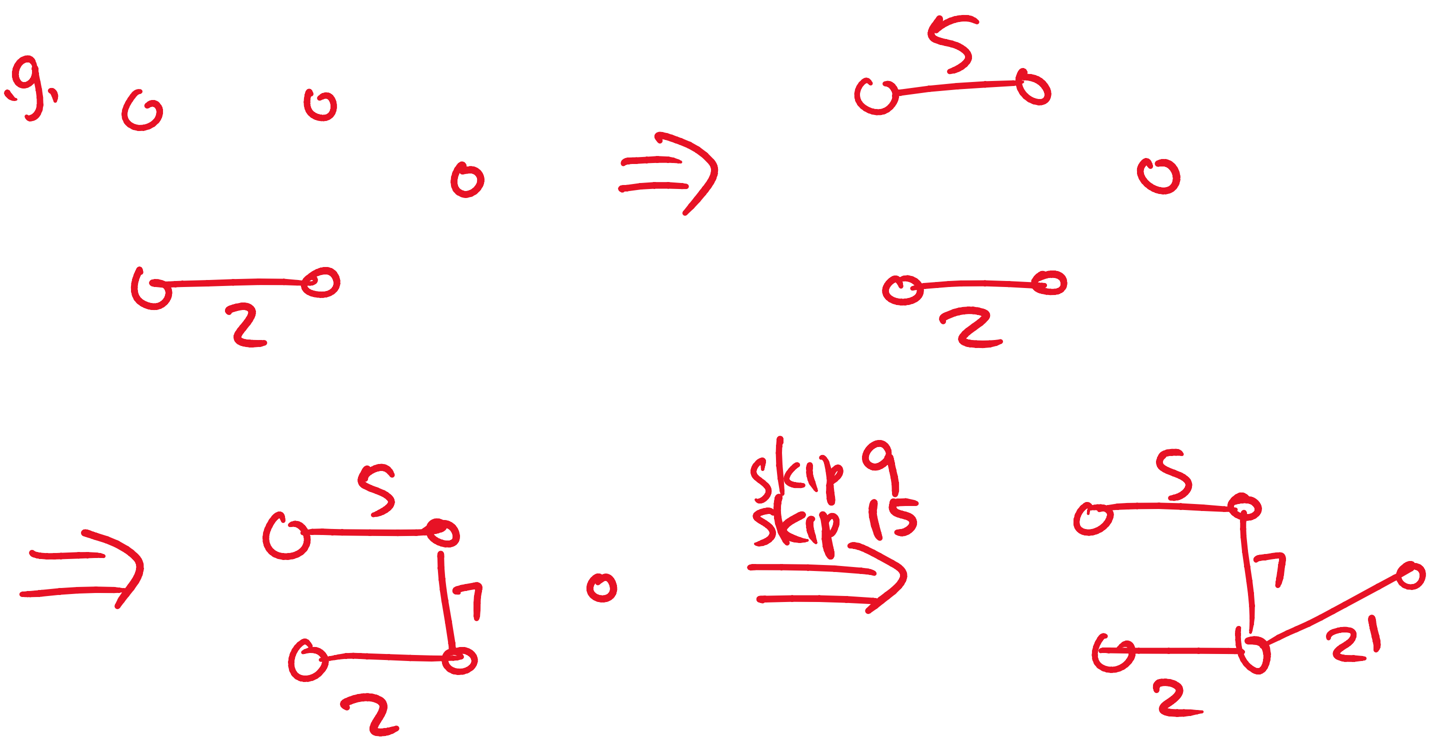
idea - greedy!

Kruskal's Alg'm (1956): High-level version

1. $T = \emptyset$
2. repeat {
3. pick next smallest-weight edge e
4. if $T \cup \{e\}$ doesn't contain a cycle
5. insert e to T

}

e.g.



Implementation: (detailed vers.)

$O(m \log n)$ time

1. Sort edges in increas. order of weight
2. Create set $\{v\}$ $\forall v \in V$

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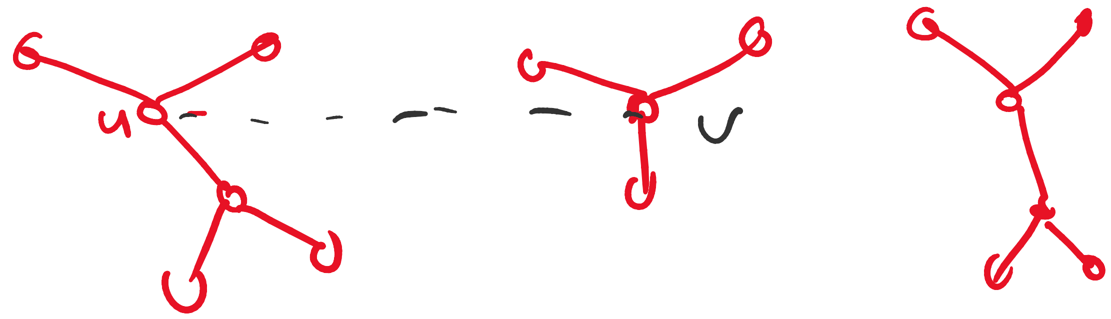
3. for each edge uv in sorted order
4.   if u & v are in different sets
5.   output uv & union the two sets

Snapshot of T:



lines 4-5: "union-find" data structure

union two disjoint sets : $O(1)$ time

find set containing v : $O(\alpha(n))$ time (amortized)

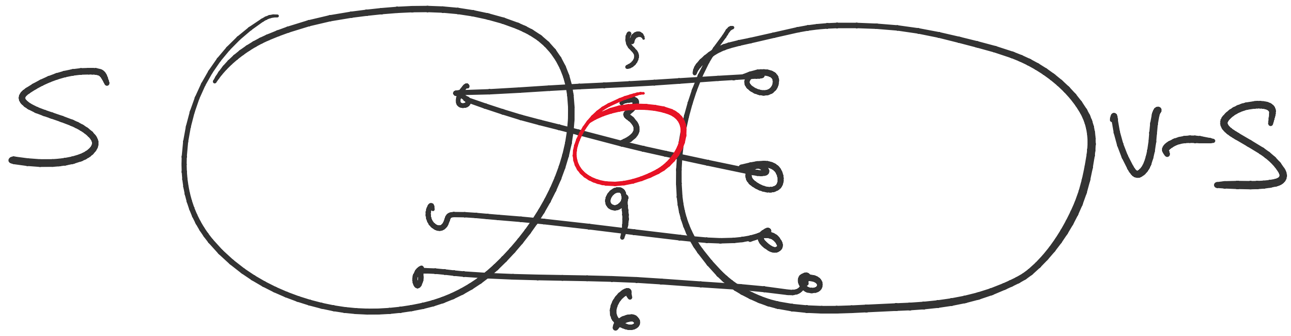
$\alpha(n) \ll \log\log\log \dots \log n$

$\Rightarrow$ $O(m\,\alpha(n))$ time

total time $\boxed{O(m \log n)}$

Correctness Pf: (assume wts are all distinct)

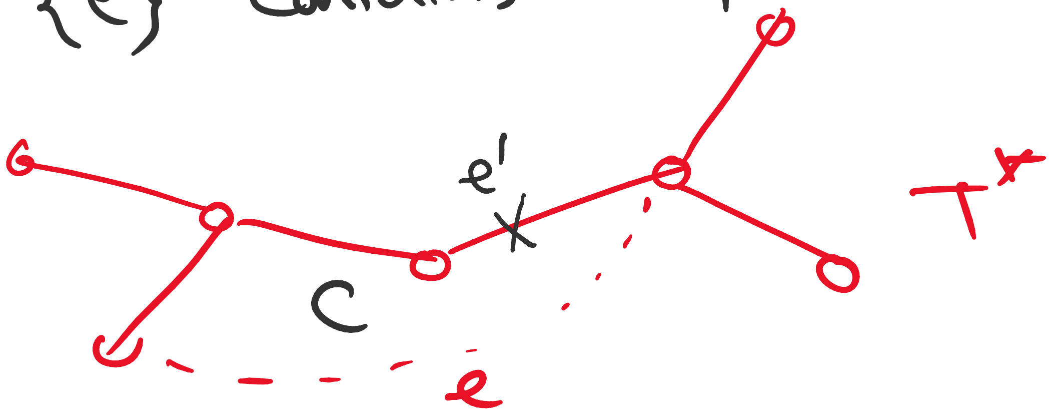
Key Lemma Given any subset $S \subseteq V$, smallest-weight edge e between S & $V-S$ must be in the MST. T^*.



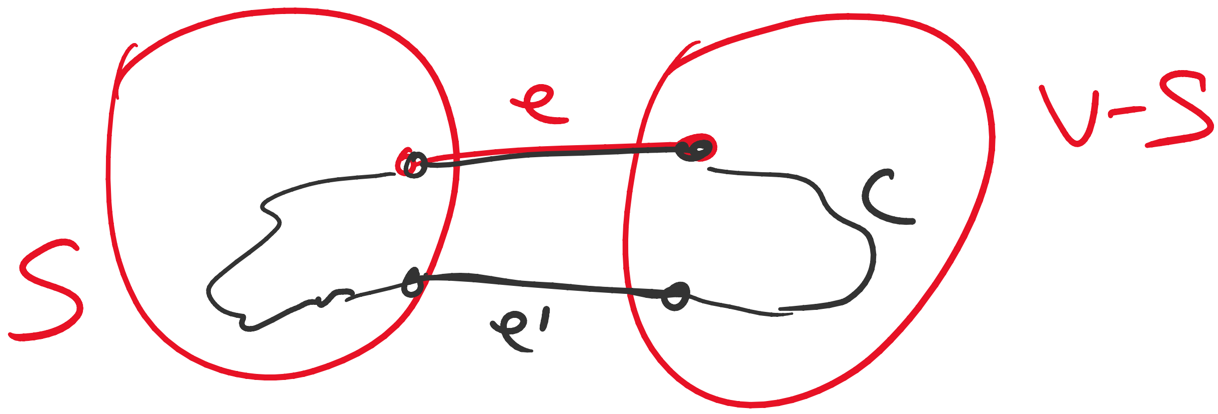
Pf: By contradiction.

Suppose $e \notin T^*$.

$T^* \cup \{e\}$ contains a cycle C.



C must contain another edge e' between S & $V-S$.



"exchange arg"

$T^* \cup \{e\} - \{e'\}$ is a tree

with weight $w(T^*) + w(e) - w(e')$

$< w(T^*)$

because $w(e) < w(e')$:

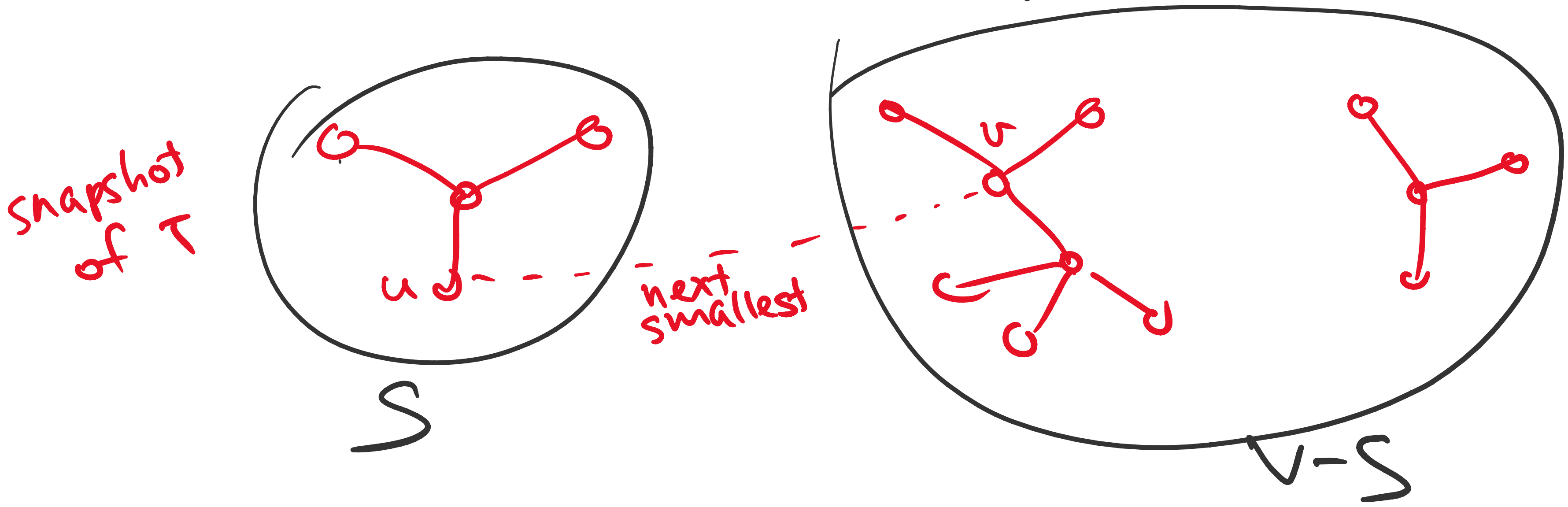
Contradiction! $\square$

Correctness Pf for Kruskal:

each edge $e = uv$ inserted to T

is in MST T^* by Key Lemma.

each edge $e = uv$ ...
is in MST T^* by Key Lemma.



Prim's Alg'm (1957): High-level vers.

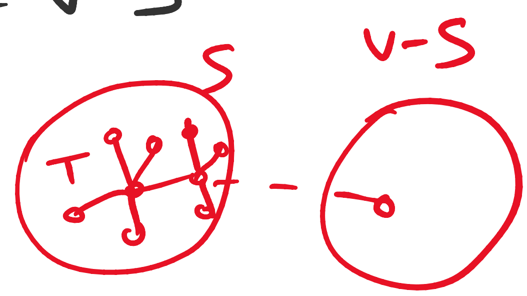
$S = \{s\}$, $T = \emptyset$

while $S \neq V$ do {

  Pick edge uv with min $w(uv)$
    with $u \in S$, $v \in V-S$

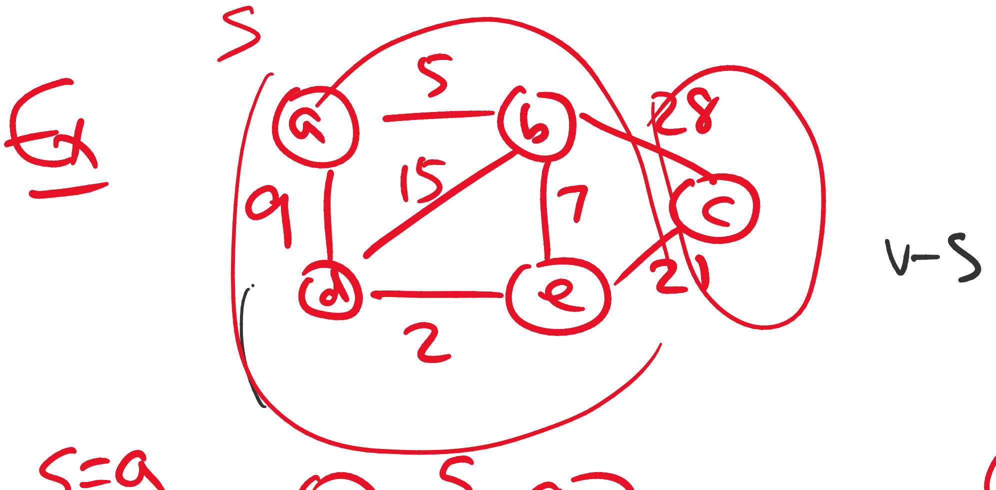
  insert uv to T,
  insert v to S

}

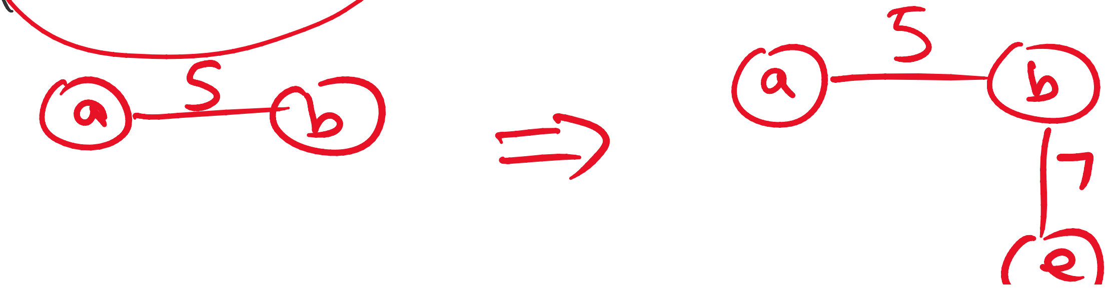


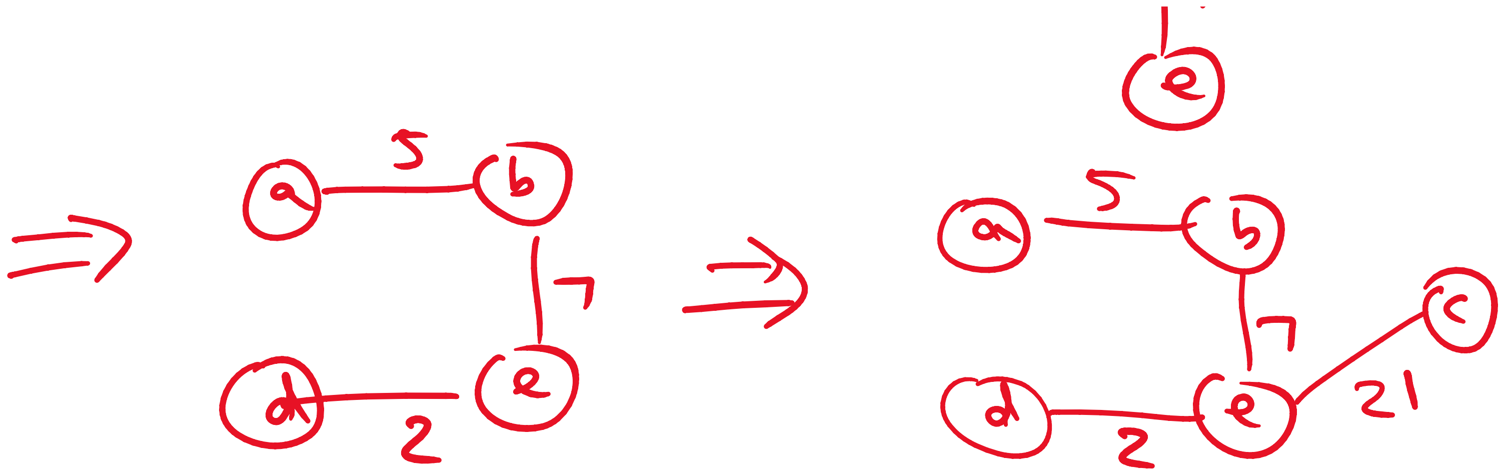
Correctness Pf: by Key Lemma again!

Ex



$s = a$





like Dijkstra,
can be implemented using Fibonacci heaps
in $\boxed{O(n \log n + m)}$ time

Other Alg'ms:

Boruvka (1926) $O(m \log n)$

⋮

Yao '75 $O(m \log\log n)$

Fredman, Tarjan '85 $O(m \log^* n)$

Gabow et al. '86 $O(m \log(\log^* n))$

Karger, Klein, Tarjan '94 $O(m)$ randomized

Chazelle '97 $O(m\,\alpha(n))$

OPEN $O(m)$ det. ?