March 28, 2019 10:58 AM

Shortest Paths (Cont'd)

Last Time:

Dijkstra's alg'm

Option 1: no data structures
$\Rightarrow O(n^2)$ time

Option 2: heap
$\Rightarrow O((m+n)\log n)$

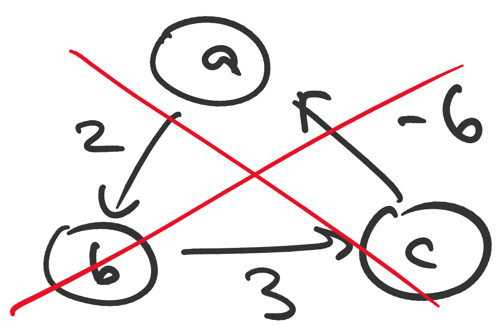
Option 3: Fibonacci heap

delete-min $O(\log n)$
~~change-key $O(\log n)$~~
decrease-key $O(1)$

$\Rightarrow O(n\log n + m \cdot 1)$
$= \boxed{O(n\log n + m)}$

Bellman-Ford Alg'm (~1956)

- solves SSSP
- neg wts allowed
  assumes no neg-wt cycles



idea - DP again!

idea - DP again!

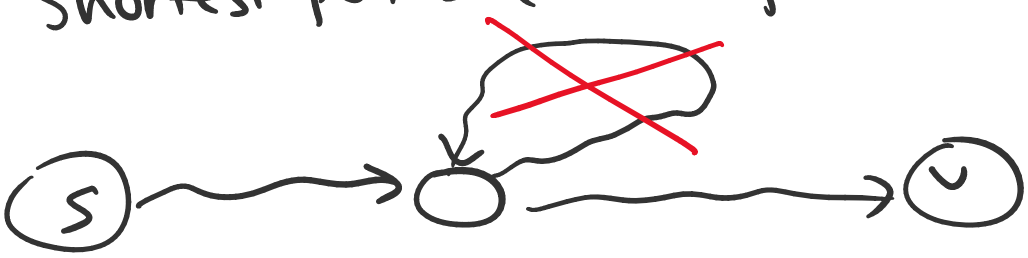
Define subproblems: $\forall v \in V,\ \forall \ell = 0, 1, \ldots, n-1$

$d(v, \ell)$ = min dist over all paths from s to v of length $\leq \ell$ (# edges)

Answer:

$d(v, n-1) \quad \forall v \in V.$

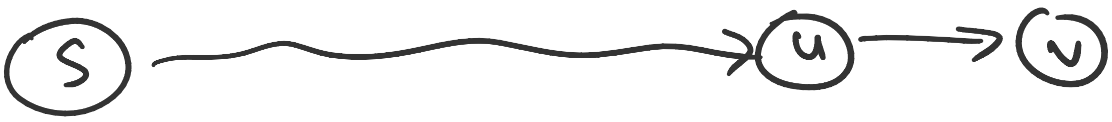
Obs shortest paths can't repeat vertices.



Base case: $d(s, 0) = 0$

$d(v, 0) = \infty \quad \forall v \in V - \{s\}$

Recursive formula:



$$\boxed{d(v, \ell) = \min_{u : (u,v) \in E} \left(d(u, \ell - 1) + w(u, v) \right)}$$

Evaluation order:

increas. ℓ.

Runtime:

$$O\Big(\underbrace{n}_{\text{choices of } \ell} \cdot \sum_{v \in V} \text{in-deg}(v) \Big)$$

$(m \geq n)$

$= \boxed{O(mn)}$ slower than Dijkstra

$$= \boxed{O(mn)}$$ slower than Dijkstra

Rmk - can be modified to detect neg-wt cycles

Obs Assume all vertices reachable from s.
Then there is no neg-wt cycle
$\iff \forall v \in V, \ d(v,n) = d(v,n-1)$.

(i.e. $\forall (u,v) \in E, \ d(u,n-1) + w(u,v) \geq d(v,n-1)$
i.e. $w(u,v) \geq d(v,n-1) - d(u,n-1)$)

(**Appl** - currency trading
$\exists$ cycle s.t. product of exchange rate > 1
Set weight $= -\log$ exchange rate)

All-Pairs Shortest Paths (APSP)

find shortest path between every pair of vertices

Method 0. run Dijkstra n times, from every start vertex s

$$\Rightarrow O(n \cdot (n \log n + m))$$
$$= \boxed{O(n^2 \log n + mn)}$$

(assume no neg wts)

Method 0'. make wts positive (Johnson's preprocessing)
then do Method 0

how? $w'(u,v) = w(u,v) + dist(s,u) - dist(s,v)$

≥ 0

computable by Bellman-Ford

$dist(s,v) \leq dist(s,u) + w(u,v)$

$$\Rightarrow \boxed{O(n^2 \log n + mn)}$$

Method 1: DP (first attempt)

Let $V = \{1, 2, \dots, n\}$.

Define subproblems:

$d(i,j,\ell)$ = min dist over all paths from i to j of length $\leq \ell$.

Formula:

$$d(i,j,\ell) = \min_k \left(d(i,k,\ell-1) + w(k,j) \right).$$

$$\Rightarrow O(n^3 \cdot n) = O(n^4)$$

Better formula:

$$i \xrightarrow{\leq \ell/2} k \xrightarrow{\leq \ell/2} j$$

$$\boxed{d(i,j,\ell) = \min_k \left(d(i,k,\ell/2) + d(k,j,\ell/2) \right)}$$

Suffices to try $\ell = 1, 2, 4, 8, \dots$

Runtime $O(n^2 \log n \cdot n)$

$= \boxed{O(n^3 \log n)}$

Method 2: Still Better DP (Floyd-Warshall '62)

Define subproblems:

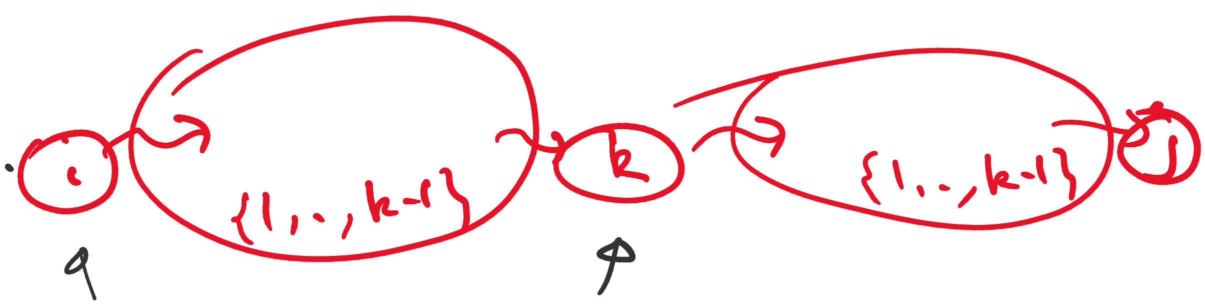
$d(i,j,k) =$ min dist over all paths from i to j s.t. all intermediate vertices are from $\{1, \ldots, k\}$

Answer: $d(i,j,n)$ $\forall i,j$

Base case ...

Formula:

$$\boxed{d(i,j,k) = \min\{\underbrace{d(i,j,k-1)}_{\text{not use } k},\ \underbrace{d(i,k,k-1) + d(k,j,k-1)}_{\text{use } k}\}}$$



Evaluation order: increas. k

Runtime: $O(n^3 \cdot 1)$

$= \boxed{O(n^3)}$

Rmk: Similar to our alg'm for DFA → reg expr

Rmk: current record for dense graphs

$\sim O(n^3/\log^{1/3} n)$ Friedman '76

$\vdots$

$O(n^3/\log n)$ C' 05

$O(n^3/\log^{5/4} n)$ Han '06

$O(n^3/\log^2 n)$ C' 07

$O(n^3/c^{\sqrt{\log n}})$ Williams '14 rand.

" C-Williams '16 det.

OPEN: $O(n^{2.99})$??