February 21, 2019 10:58 AM

Divide & Conquer

Problem 1: Multiplying LARGE numbers

Given 2 n-bit numbers $A = a_{n-1} a_{n-2} \cdots a_0$
$B = b_{n-1} b_{n-2} \cdots b_0$,

compute $AB = c_{2n-1} c_{2n-2} \cdots c_0$

"Elementary School" Alg'm: (?? B.C.)

```
     1011
  x  1101
 --------
     1011
   1011
  1011
 --------
 10001111
```

n shifts
n additions
& each $\Theta(n)$ time

$\Rightarrow$ total $\boxed{\Theta(n^2)}$ time

Surprisingly, can do better!

Karatsuba's Alg'm: (1962)

approach - divide $A = \underbrace{a_{n-1} \cdots a_{n/2}}_{A'} \; \underbrace{a_{n/2-1} \cdots a_0}_{A''}$

$B = \underbrace{b_{n-1} \cdots b_{n/2}}_{B'} \; \underbrace{b_{n/2-1} \cdots b_0}_{B''}$

e.g. $A = 1011 \rightarrow 11$
$A' = 10 \rightarrow 2$
$A'' = 11 \rightarrow 3$

$A'' = 11 \to 3$ B' B''

$$A = A' 2^{n/2} + A''$$
$$B = B' 2^{n/2} + B''$$

first idea:

$$AB = (A' 2^{n/2} + A'')(B' 2^{n/2} + B'')$$
$$= A'B' 2^n + \boxed{(A'B'' + A''B')} 2^{n/2} + A''B''$$

(2 shifts, 3 adds, 4 mults of $(n/2)$-bit #s) by recursion

$$\Rightarrow T(n) = \begin{cases} O(1) & \text{if } n \le \text{const} \\ 4T(\frac{n}{2}) + O(n) & \text{else} \end{cases}$$

["Master" Thm: $T(n) = aT(\frac{n}{b}) + n^d$
$\Rightarrow O(n^{\log_b a})$ if $d < \log_b a$]

$$\Rightarrow O(n^{\log_2 4}) = \boxed{O(n^2)}$$

no improvement :-(

more clever idea: rewrite

$$A'B'' + A''B' = (A' + A'')(B' + B'') - A'B' - A''B''$$

mult(A, B):
  if $n \le$ const ...

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divide A into A', A''

B into B', B''

$C_1 = \text{mult}(A', B')$

$C_2 = \text{mult}(A'', B'')$

$C_3 = \text{mult}(A' + A'', B' + B'')$

return $C_1 2^n + (C_3 - C_1 - C_2) 2^{n/2} + C_2$

$$\Rightarrow \quad T(n) = 3\,T\left(\frac{n}{2}\right) + O(n)$$

$$\Rightarrow \quad O\left(n^{\log_2 3}\right) = \boxed{O\left(n^{1.59}\right)}$$

faster!

Rmk. can be further refined

$$T(n) = 5\,T\left(\frac{n}{3}\right) + O(n) \Rightarrow O\left(n^{1.47}\right)$$

$$T(n) = 7\,T\left(\frac{n}{4}\right) + O(n) \Rightarrow O\left(n^{1.41}\right)$$

$\vdots$

$O(n^{1+\varepsilon})$ for any const $\varepsilon > 0$ (Toom-Cook '63)

$O(n \log n \log\log n)$ (Schönhage-Strassen '71)

$O(n \log n\, \log\log\log\log \cdots \log n)$ (Fürer '07)

$O(n \log n)$? OPEN!!

Problem 2: Selection

Given n numbers $a_1, \dots, a_n$, & k, (unsorted)

find the k^{th} smallest

e.g. 50, 82, 43, 19, 96, 32, 74, 25

$k=4$: 43

e.g. $k = \lfloor n/2 \rfloor$

$\Rightarrow$ median

Alg'm 0. Sort & look up answer

$\Rightarrow$ $O(n \log n)$ time

Alg'm 1 Selectionsort variant

$\Rightarrow$ $O(kn)$ time

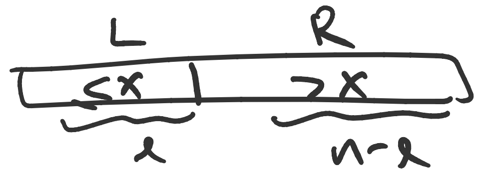
Alg'm 2 heapsort variant

$\Rightarrow$ $O(\underbrace{n}_{\text{build heap}} + \underbrace{k \log n}_{k \text{ delete-min}})$ time

is $O(n)$ possible for any k?

Alg'm 3 quicksort variant

select($\{a_1, .., a_n\}$, k):



1. if $n=1$ return a_1

→ 2. pick "pivot" x how?

3. split into $L = \{a_i : a_i \le x\}$, $\ell = |L|$

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$R = \{a_i : a_i > x\}$

$O(n)$ time $\Rightarrow$

4. if $k \le \ell$ then
return Select(L, k)
else return Select$(R, k-\ell)$

$\max\{T(\ell), T(n-\ell)\} \Rightarrow$

$$T(n) \le \max\{T(\ell), T(n-\ell)\} + O(n)$$

ideal case: $\ell = n/2$

$$T(n) = T\left(\frac{n}{2}\right) + O(n)$$
$$\Rightarrow O\left(n + \frac{n}{2} + \frac{n}{4} + \cdots\right)$$
$$= O(n).$$

bad case: $\ell = 1$ or $\ell = n-1$

$$T(n) = T(n-1) + O(n)$$
$$\Rightarrow O(n + n-1 + n-2 + \cdots)$$
$$= O(n^2).$$

(can give $O(n)$ expected time by randomly picking pivot)

Alg'm by Blum, Floyd, Rivest, Pratt, Tarjan (1973)

clever idea: pick a good pivot x close to the median
by taking median of medians of 5

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a recursive call

Replace line 2 by:

2.1. split $\{a_1, \dots, a_n\}$ into groups $G_1, \dots, G_{n/5}$ of 5 each

$O(n)$ time 2.2. for $i = 1$ to $n/5$ do x_i = median of G_i

$T(\frac{n}{5})$ 2.3. $x = \text{select}(\{x_1, \dots, x_{n/5}\}, \frac{n}{10})$

e.g. $\underbrace{1, 10, 5, 8, 21}_{G_1}, \underbrace{34, 6, 7, 12, 23}_{G_2}, \underbrace{2, 4, 30, 11, 25}_{G_3}$

$x_1 = 8$ $x_2 = 12$ $x_3 = 11$

$x = 11$

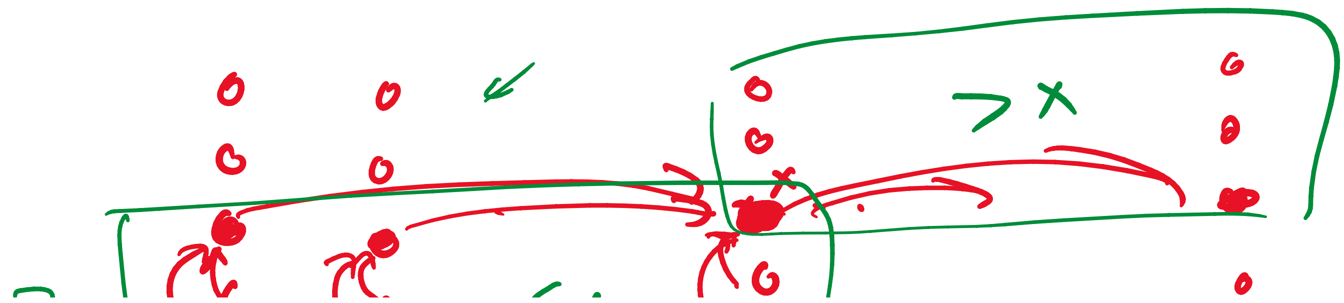
Lemma $\frac{3n}{10} \lesssim \ell \lesssim \frac{7n}{10}$

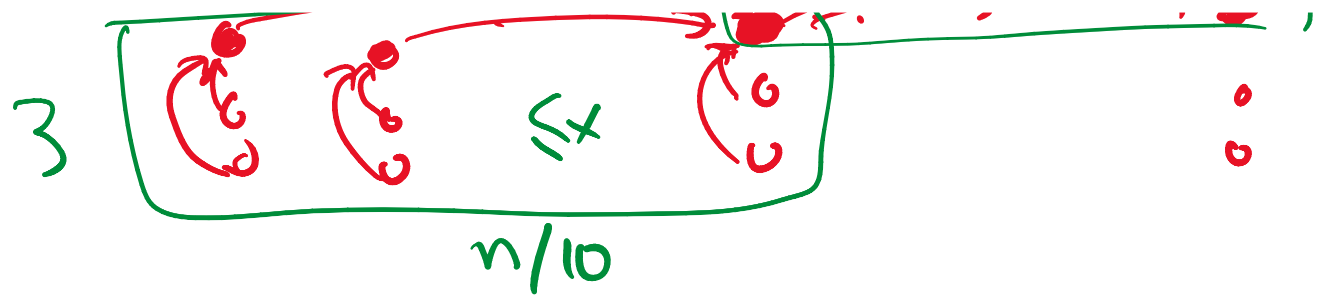
Pf: $\sim \underbrace{\frac{n}{10}}$ groups G_i with $x_i \le x$

each such group has $\underline{3}$ elements $\underline{\le} x_i$

$\Longrightarrow \gtrsim 3 \cdot \frac{n}{10}$ elements $\le x$

By symmetry, $\gtrsim \frac{3n}{10}$ elements $> x$. □





$$\Rightarrow \quad T(n) \le \max\left(T(\ell), T(n-\ell)\right) + \underline{T\left(\tfrac{n}{5}\right)} + O(n)$$

$$T(n) \le T\left(\frac{7n}{10}\right) + T\left(\frac{n}{5}\right) + O(n)$$

$$\Rightarrow \quad T(n) = \boxed{O(n)}$$

e.g. guess & verify
like HW1 Q3

because $\frac{7}{10} + \frac{1}{5} = \frac{9}{10} < 1$