January 15, 2019 2:10 AM

# CS 374: Algorithms & Models of Computation

## Goal.

techniques to design alg'ms

- how to solve problems on computers (efficiently)

need math defn/model of computation

- understand what problems can or can't be solved

prove mathematically

## Course Overview

### I. Models of Computation

finite automata ⟺ regular expr
context free grammars
Turing machines

### II. Algorithms Design

divide & conquer
dynamic Programming
greedy

### III. Undecidability & NP-Completeness

**Ex1** Given n numbers, can you find 3 summing exactly to 100?

(3SUM)

e.g. 82, 43, 19, 96, 32, 74, 25

yes

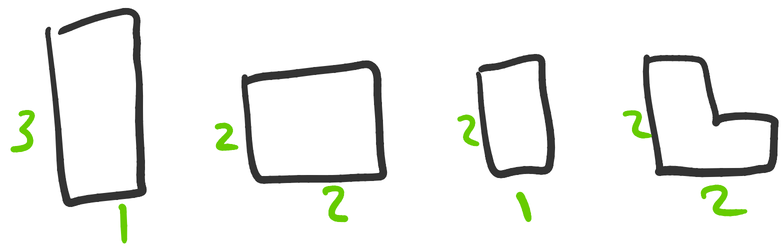
brute force: $O(n^3)$ time — yes

cleverer alg'm: $O(n^2)$

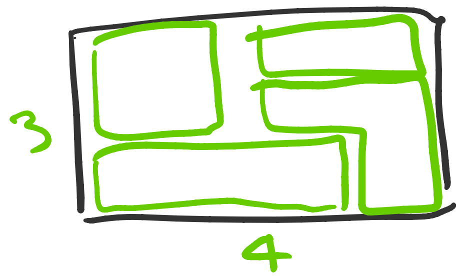
fastest alg'm: still open!

[current record: about $O\left(\frac{n^2}{\log^2 n}\right)$ by C. '17]

**Ex 2** Given $n$ polygons & a box, can you pack them in box?

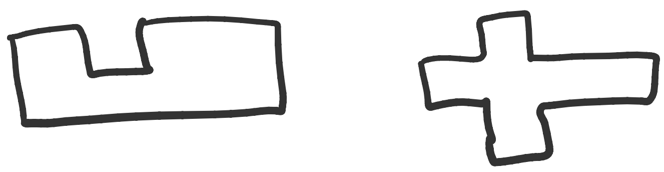



$\Rightarrow$




NP-complete

**Ex 3** Given $n$ polygons, can you tile the entire plane?



undecidable

no alg'm is possible!

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# PART I: MODELS OF COMPUTATION

## Math Prelims

### Strings

# Strings

A **string** is a finite sequence of symbols from a finite set $\Sigma$ ← called alphabet

e.g. strings over $\Sigma = \{0,1\}$
0110, 01, 0

Empty string is denoted $\varepsilon$.

Let $\Sigma^* = \{$ all strings over $\Sigma\}$.

Let $x, y$ be strings.

a) length $|x|$

e.g. $|0110| = 4$, $|01| = 2$, $|\varepsilon| = 0$

b) concatenation $xy$

e.g. $x = 01$, $y = 101$
$\Rightarrow xy = 01101$

$$(xy)z = x(yz)$$
$$|xy| = |x| + |y|$$
$$\varepsilon x = x\varepsilon = x$$

c) $i^{th}$ power $x^i = \underbrace{x \cdots x}_{i \text{ times}}$

e.g. $(101)^3 = 101101101$

$x^0 = \varepsilon$

d) $x$ is a substring of $y$ if $y = wxz$ for some strings $w, z$

( prefix if $w = \varepsilon$, suffix if $z = \varepsilon$)

e) other ops:

$x^R$ = reverse of $x$

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(can be defined recursively:

$$x^R = \begin{cases} y^R a & \text{if } x = ay \\ & a \in \Sigma, \\ & y \in \Sigma^* \\ \varepsilon & \text{if } x = \varepsilon \end{cases}$$
)

$$(xy)^R = y^R x^R$$

## Languages

A <u>language</u> is a set of strings
(i.e. subset of $\Sigma^*$)

e.g. $\{0110, 01, 0\}$ is a lang.

$\{x \in \{0,1\}^* \mid |x| \text{ is even}\}$

{all words in English dictionary}
(over $\Sigma = \{'a', \ldots, 'z'\}$)

finite, boring

{all syntactically valid Java programs}
more interesting

{all prime numbers in binary}

⋮

(decision problems can be encoded as languages)

Let $L_1, L_2$ be languages.

a) union $L_1 \cup L_2$

intersection $L_1 \cap L_2$

complement $\overline{L_1} = \Sigma^* \setminus L_1$

difference $L_1 \setminus L_2 = L_1 \cap \overline{L_2}$

b) <u>concatenation</u>

$$L_1L_2 = \{ xy \mid x \in L_1, y \in L_2 \}$$

e.g. $L_1 = \{0, 00\}$, $L_2 = \{1, 01\}$

$\Rightarrow L_1L_2 = \{01, 001, 0001\}$

e.g. $L_1 = \{0, 00, 000, \ldots\}$

$L_2 = \{1, 11, 111, \ldots\}$

$\Rightarrow L_1L_2 = \{ 0^i 1^j \mid i,j \geq 1 \}$

c) <u>$i^{th}$ power</u> $L^i = \underbrace{LL \cdots L}_{i \text{ times}}$

e.g. $\{1, 01\}^2 = \{\underline{11, 101, 011, 0101}\}$

$L^0 = \{\varepsilon\}$.

d) <u>Kleene star</u>

$$L^* = \bigcup_{i=0}^{\infty} L^i$$

$$= L^0 \cup L^1 \cup L^2 \cup \cdots$$

e.g. $\{01\}^* = \{\varepsilon, 01, 0101, 010101, \ldots\}$

$\{0,1\}^*$ as defined before

$\{1, 01\}^* = \{\varepsilon, 1, 01,$
$11, 101, 011, 0101,$
$111, 1101, \ldots\}$

$= \{ x \in \{0,1\}^* \mid x$ does not containing 00 as a substring & x ends in 1 $\}$

containing ... }

& $x$ ends in 1 }
(unless $x = \varepsilon$)

$\{00, 01, 10, 11\}^* = \{x \in \{0,1\}^* \mid |x| \text{ even}\}$

$\{00, 000, 0000, \dots\}^*$
$= \{\varepsilon, 00, 000, 0000, \dots\}$

⋮

e) other ops: $L^+ = \bigcup_{i=1}^{\infty} L^i$

$\{0,1\}^+ = \{\text{all nonempty strings over } \{0,1\}\}$

$$L^+ = LL^* = L^*L = \begin{cases} L^* \setminus \{\varepsilon\} & \text{if } \varepsilon \notin L \\ L^* & \text{else} \end{cases}$$

Rmk: there are countably many strings (over finite $\Sigma$)

there are uncountably many langs. (but countably many Java programs)

$\Rightarrow \exists$ langs that can't be "solved" !!