CS 374 Lab 17: Graph Decomposition

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The starting point for understanding connectivity in undirected graphs is to consider the connected components which can be found in linear time. In directed graphs the starting point is a bit more involved. We need to consider the strong connected components and the meta-graph, which is a DAG, obtained by shrinking each strong connected component into a single node and removing all parallel edges. Obtaining this representation in linear time requires much more work but we can assume that algorithm as a black box. Several basic question on directed graph connectivity rely on the meta-graph and its properties.

1. There are n light bulbs in a garden. These bulbs can be turned on manually by flipping on the switches at the light posts. Also, each light post can broadcast turn-on signals to some other pre-defined light posts in the garden, turning them on. When a light post is turned on, it will automatically broadcast a turn-on signal to its pre-defined light posts.

   This signal broadcasting is directional. If a broadcasts to b, it is not necessarily true that b also broadcasts to a.

   So one can manually flip on some of the switches to the light posts, and those light posts will broadcast a turn-on signal to other light posts. These will in turn be switched on and broadcast signals to their own pre-defined set of light posts, and so on.

   Given each light post in the garden and the respective light posts to which they broadcast, derive a linear time algorithm for finding the minimum number of switches needed to be flipped to light up the whole garden. (Linear time means $O(n + m)$ where n is the number of light posts and m is the number of broadcast associations between them).

   Source: *ACM ICPC 2010 World Finals Warmup 2*

   Example Case: Number of lights : 5 , Number of broadcast associations: 4
   Associations : 1=>2, 1=>3 3=>4, 5=>3
   Answer: Minimum number of flips required : 2 ,
   Turning on switches 1 and 5 should light up the whole garden

   Hints:
   (a) Model the problem using directed graphs.
   (b) What is the solution if the graph in question is strongly connected?
   (c) What is the solution if the graph in question is a DAG?
   (d) What is the solution in general?

2. Let G be a directed acyclic graph. Prove that G has a unique topological sort if and only if it has a Hamiltonian Path.