Shortest paths

$G=(V,E)$ directed

$w: E \to \mathbb{R}$ no negative cycles

source vertex s

Compute tree of shortest paths from s to V

Unweighted: BFS $O(V+E)$

Weighted, positive: Dijkstra $O(E \log V)$

Weighted, dag: DFS/dyn prog $O(V+E)$

cycles, neg edges — Now what?

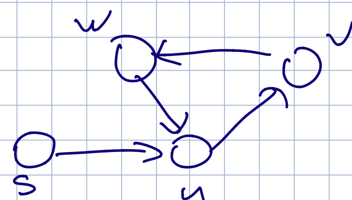
$\text{dist}(v)$ = shortest-path dist from s to v

$$\text{dist}(v) = \begin{cases} 0 & s=v \\ \min\{\text{dist}(u) + \ell(u \to v) \mid u \to v\} & \text{otherwise} \end{cases}$$

Memoize into G

~~Order?~~

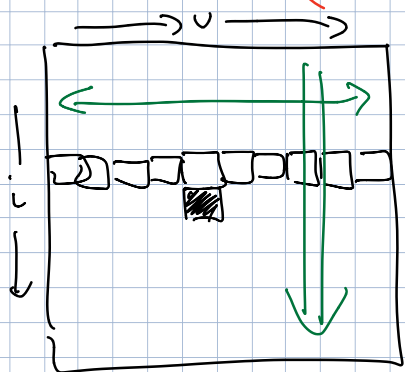
BAD!



we need another parameter

$\text{dist}(i,v)$ = length of shortest path from s to v $\boxed{\text{with} \le i \text{ edges}}$

$$\mathrm{dist}(i,v) = \begin{cases} 0 & \text{if } v = s \\ \infty & \text{if } i = 0 \text{ and } v \neq s \\ \min\left\{ \begin{array}{l} \min\{\mathrm{dist}(i-1,u) + \ell(u\to v) \mid u \to v\} \\ \mathrm{dist}(i-1,v) \end{array} \right\} & \text{otherwise} \end{cases}$$



Top: $\mathrm{dist}(V-1, v)$

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BellmanFord(V, E, ℓ):
    dist[s] ← 0                      pred[s] ← NULL
    for all v ≠ s
        dist[v] ← ∞                  pred[v] ← NULL
    for i ← 1 to V-1                 O(V)×
        for all edges u→v            O(E)
            if u→v is tense
                relax u→v
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After i^{th} iteration of main loop $\mathrm{dist}[v] \leq$ Bellman's $\mathrm{dist}(i,v)$

$\boxed{O(VE) \text{ time!}}$

All-pairs shortest paths.

Compute $dist[u,v]$ for all $u, v \in V$

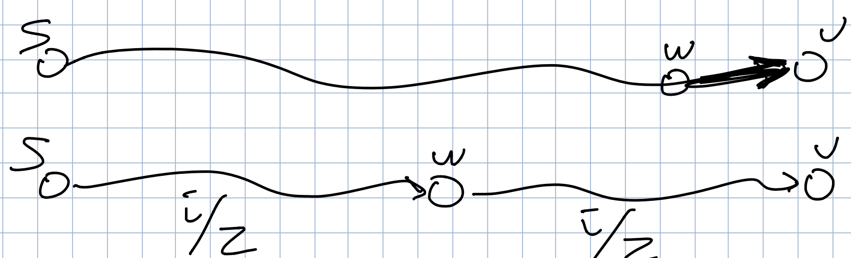
$\quad\llcorner$ shortest path dist from u to v

$$\boxed{\begin{array}{l}\text{For all verts } s \\ \quad dist[s, \ldots] \leftarrow \text{BellmanFord}(s)\end{array}}$$

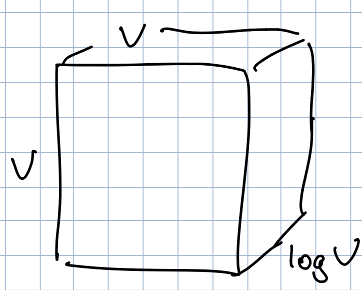
$O(V^2E)$ time

$dist[i,u,v]$ = length of sh. path $u \rightsquigarrow v$ with $\leq i$ edges

$$dist[i,u,v] = \begin{cases} \vdots \\ \min\left\{ dist[i-1,u,w] + \ell(w \to v) \;\middle|\; w \in V \right\} \\ \vdots \end{cases}$$



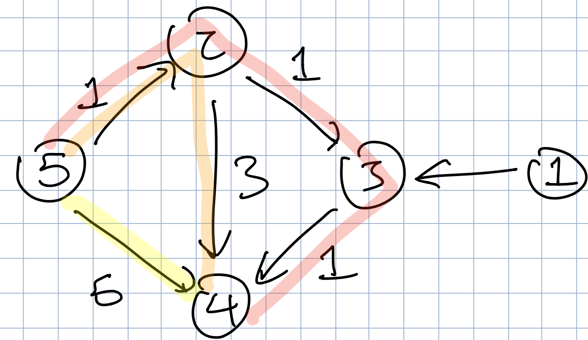
$$dist[i,u,v] = \begin{cases} \vdots \\ \min\left\{ dist[i/2,u,w] + dist[i/2,w,v] \;\middle|\; w \in V \right\} \\ \vdots \end{cases}$$

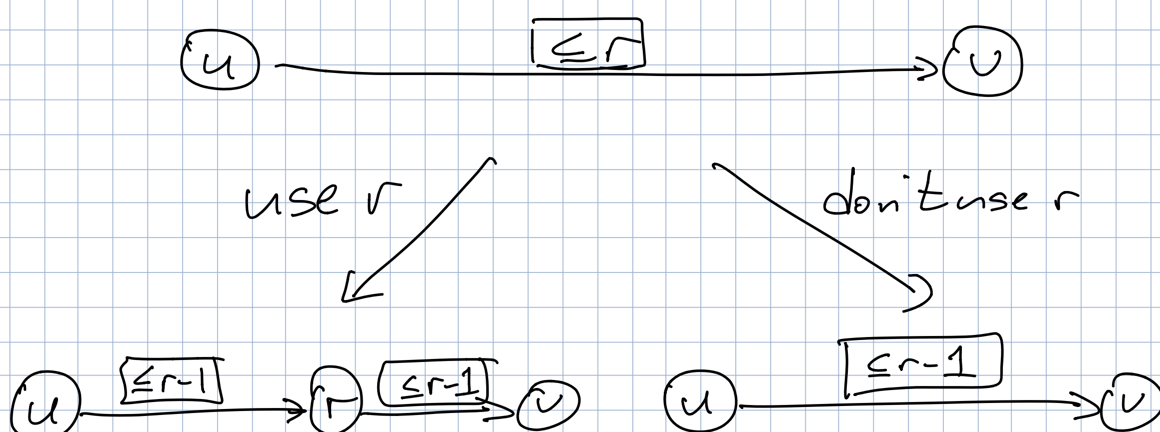


$\boxed{O(V^3 \log V)}$

vs. $O(V^2E)$

dist'[r, u, v] = length of sh. path $u \rightsquigarrow v$ where every intermediate vertex has index $\le r$.





$$\text{dist}[r,u,v] = \min\left\{ \begin{array}{l} \text{dist}[r-1,u,v] \\ \text{dist}[r-1,u,r] + \text{dist}[r-1,r,v] \end{array} \right\}$$

$\underline{O(V^3)}$ time