

Recursion

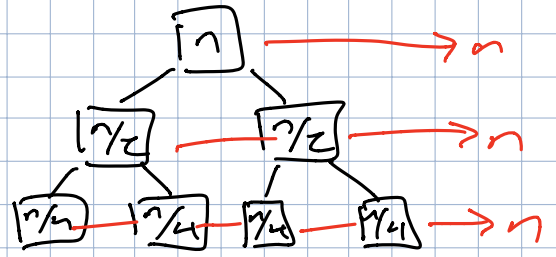
Reduce given instance of some problem to smaller instances of same problem

Merge Sort

if n is big enough
 reduce to sort(Left half)
 sort(Right half)
 then Merge

else
 brute force

$$T(n) = 2T(n/2) + O(n)$$



Multiplication

lattice: $O(n^2)$

Egyptian / Russian / Ethiopian

→ Duplation + Mediation

x	y	prod
123	456	0
61	912	456
30	1824	1368
⋮	⋮	⋮
⋮	⋮	⋮

Mult(x,y):

```

prod ← 0
while (x > 0)
  if x is odd
    prod ← prod + y · (x mod 10)
    x ← x - (x mod 10) (x mod 10)
  x ← x / 10 10
  y ← y y · 10
return prod
  
```

Invariant: $(xy + prod)$ doesn't change

$$xy = \begin{cases} 0 & \text{if } x=0 \\ \frac{x}{2} \cdot 2y & \text{if } x \text{ even and } x > 0 \\ \lfloor \frac{x}{2} \rfloor \cdot 2y + y & \text{if } x \text{ odd} \end{cases}$$

Time: $T(m,n) = \text{Time to multiply } m\text{-bits} \times n\text{-bits}$

$$T(m,n) = T(m-1, n+1) + O(m+n)$$

$$T(m,n) = O(m^2 + mn)$$

Kolmogorov "n² conjecture" 1956

23-yr-old Anatoliĭ Karatsuba 1960

NOPE

$$x = \boxed{a} \boxed{b} = a \cdot 10^m + b$$

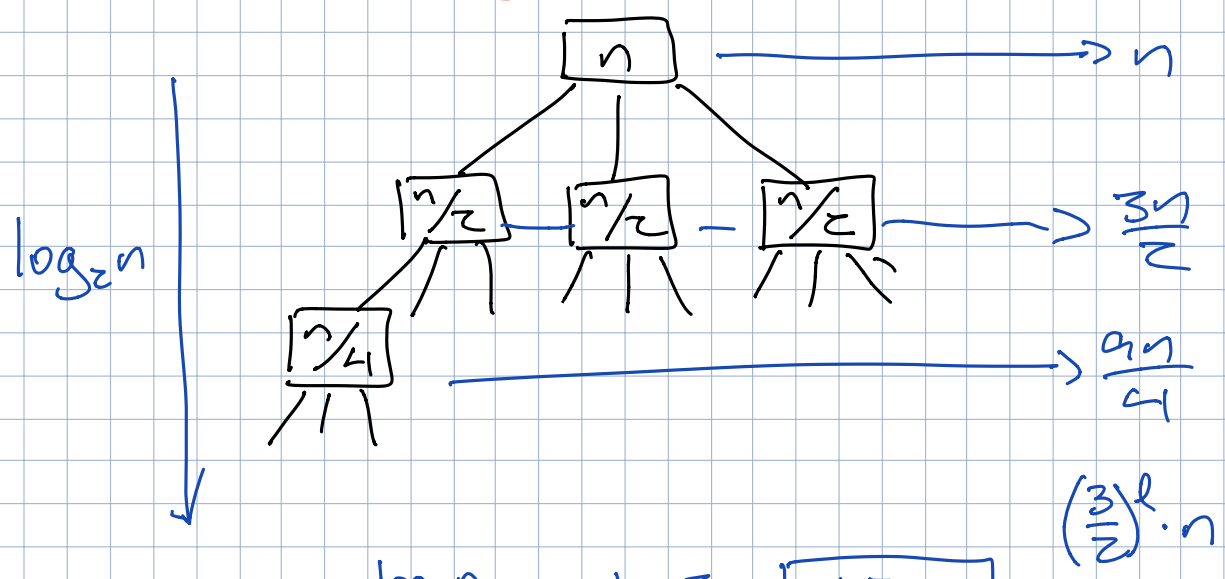
$$y = \boxed{c} \boxed{d} = c \cdot 10^m + d$$

$$xy = \underline{ac} 10^{2m} + (bc + ad) 10^m + \underline{bd}$$

$$(a-b)(c-d) = \underline{ac} - (bc + ad) + \underline{bd} \quad \leftarrow \text{Gauss 1840}$$

$$T(n) = 4T(n/2) + O(n) \quad \longrightarrow \quad O(n^2)$$

$$T(n) = 3T(n/2) + O(n)$$



$$\# \text{leaves} = 3^{\log_2 n} = n^{\log_2 3} = \boxed{n^{1.7\dots}}$$

Selection

Given unsorted array $A[1..n]$ Find ~~median~~ k th smallest element and its index

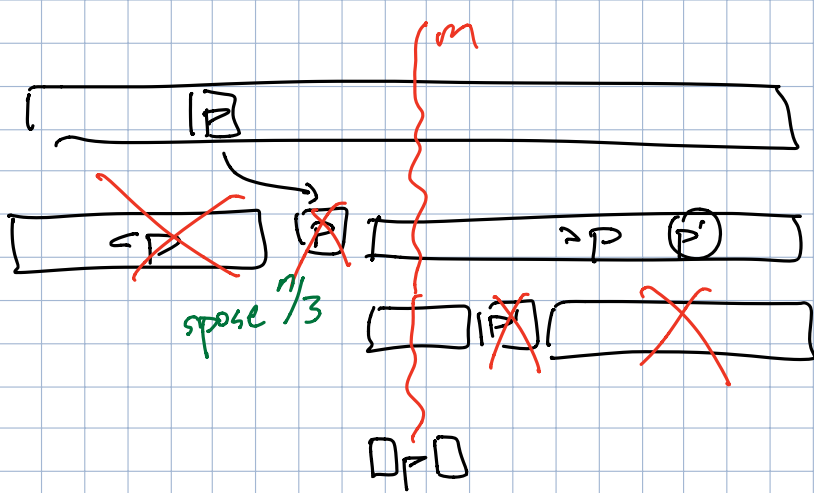
Sorting: $O(n \log n)$ time

$O(n)$ time!



Blum Floyd Pratt Rivest Tarjan 1970

"One-armed quicksort"



Select(A[1..n], k):

if $n < 100$ brute force

else

$p \leftarrow$ choose recursively!

$p \leftarrow$ PARTITION(A, p)

if $p = k$ return A[k]

if $p < k$

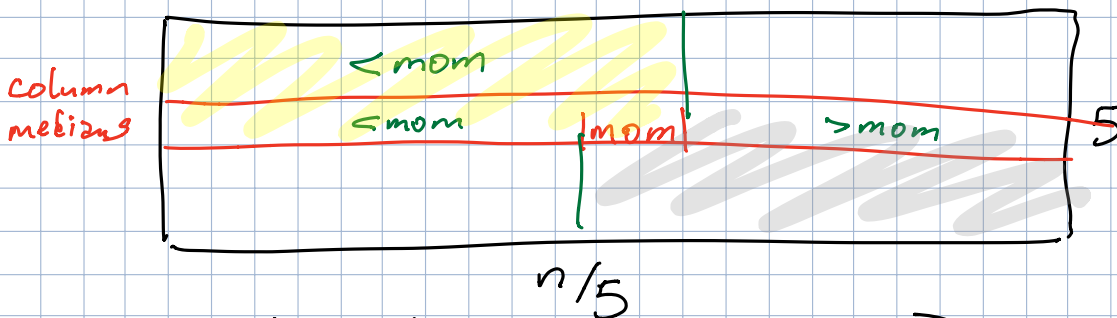
return Select(A[p+1..n], k-p)

if $p > k$

return Select(A[1..p-1], k)

$$T(n) \leq O(n) + \max_{1 \leq p \leq n} \{T(p-1), T(n-p)\}$$

$$\leq O(n) + T(n-1) = \underline{O(n^2)}$$



For $j = 1$ to $n/5$

$A[3, j] \leftarrow$ median of col j

$\} O(n)$

$mom \leftarrow$ median(A[3, *])

recursion!

$$\frac{3n}{10} < \text{rank}(mom) < \frac{7n}{10}$$

$$T(n) \leq O(n) + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right)$$

