# Greedy Algorithms

Lecture 19

April 4, 2017

# Part I

## Greedy Algorithms: Tools and Techniques

# What is a Greedy Algorithm?

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Greedy algorithms:

1. make decision incrementally in small steps *without backtracking*
2. decision at each step is based on improving *local or current* state in a myopic fashion without paying attention to the *global* situation
3. decisions often based on some fixed and simple *priority* rules

# Pros and Cons of Greedy Algorithms

Pros:

1. Usually (too) easy to design greedy algorithms
2. Easy to implement and often run fast since they are simple
3. Several important cases where they are effective/optimal
4. Lead to a first-cut heuristic when problem not well understood

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2. Many greedy algorithms possible for a problem and no structured way to find effective ones

CS 374: Every greedy algorithm needs a proof of correctness

# Greedy Algorithm Types

Crude classification:

1. Non-adaptive: fix some ordering of decisions a priori and stick with the order
2. Adaptive: make decisions adaptively but greedily/locally at each step

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Plan:

1. See several examples
2. Pick up some proof techniques

# Part II

# Scheduling Jobs to Minimize Average Waiting Time

# The Problem

- $n$ jobs $J_1, J_2, \ldots, J_n$. $J_i$ has non-negative processing time $p_i$
- One server/machine/person available to process jobs.
- Schedule/order the jobs to minimize total or average *waiting time*
- Waiting time of $J_i$ in schedule $\sigma$: sum of processing times of all jobs scheduled before $J_i$

| | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ | $J_6$ |
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| ***time*** | 3 | 4 | 1 | 8 | 2 | 6 |

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**Optimal schedule:**

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**Optimal schedule:** Shortest Job First. $J_3, J_5, J_1, J_2, J_6, J_4$.

# Optimality of SJF

## Theorem

*Shortest Job First gives an optimum schedule for the problem of minimizing total waiting time.*

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Assume without loss of generality that job sorted in increasing order of processing time and hence $p_1 \leq p_2 \leq \ldots \leq p_n$ and SJF order is $J_1, J_2, \ldots, J_n$.

# Inversions

## Definition

A schedule $J_{i_1}, J_{i_2}, \ldots, J_{i_n}$ is said to have an inversion if there are jobs $J_a$ and $J_b$ such that $S$ schedules $J_a$ before $J_b$, but $p_a > p_b$.

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## Claim

*If a schedule has an inversion then there is an inversion between two* adjacently *scheduled jobs.*

Proof: exercise.

# Proof of optimality of SJF

Recall SJF order is $J_1, J_2, \ldots, J_n$.

- Let $J_{i_1}, J_{i_2}, \ldots, J_{i_n}$ be an optimum schedule with fewest inversions.
- If schedule has no inversions then it is identical to SJF schedule and we are done.
- Otherwise there is an $1 \leq \ell < n$ such that $i_\ell > i_{\ell+1}$ since schedule has inversion among two adjacently scheduled jobs

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## Claim

*The schedule obtained from $J_{i_1}, J_{i_2}, \ldots, J_{i_n}$ by exchanging/swapping positions of jobs $J_{i_\ell}$ and $J_{i_{\ell+1}}$ is also optimal and has one fewer inversion.*

Assuming claim we obtain a contradiction and hence optimum schedule with fewest inversions must be the SJF schedule.

# Part III

## Scheduling to Minimize Lateness

# Scheduling to Minimize Lateness

1. Given jobs $J_1, J_2, \ldots, J_n$ with deadlines and processing times to be scheduled on a single resource.
2. If a job $i$ starts at time $s_i$ then it will finish at time $f_i = s_i + t_i$, where $t_i$ is its processing time. $d_i$: deadline.
3. The lateness of a job is $l_i = \max(0, f_i - d_i)$.
4. Schedule all jobs such that $L = \max l_i$ is minimized.

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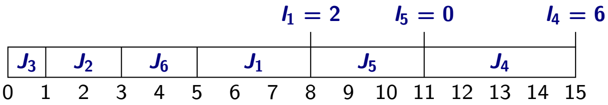
|---|---|---|---|---|---|---|
| $t_i$ | 3 | 2 | 1 | 4 | 3 | 2 |
| $d_i$ | 6 | 8 | 9 | 9 | 14 | 15 |



# Greedy Template

```
Initially R is the set of all requests
curr_time = 0
max_lateness = 0
while R is not empty do
    choose i ∈ R
    curr_time = curr_time + t_i
    if (curr_time > d_i) then
        max_lateness = max(curr_time − d_i, max_lateness)

return max_lateness
```

Main task: Decide the order in which to process jobs in $R$

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# Three Algorithms

1. Shortest job first — sort according to $t_i$.
2. Shortest slack first — sort according to $d_i - t_i$.
3. $\mathrm{EDF}$ = Earliest deadline first — sort according to $d_i$.

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3. EDF = Earliest deadline first — sort according to $d_i$.

Counter examples for first two: exercise

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**Theorem**

*Greedy with* $\mathrm{EDF}$ *rule minimizes maximum lateness.*

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*Greedy with* EDF *rule minimizes maximum lateness.*

Proof via an exchange argument.

Idle time: time during which machine is not working.

**Lemma**

*If there is a feasible schedule then there is one with no idle time before all jobs are finished.*

# Inversions

Assume jobs are sorted such that $d_1 \leq d_2 \leq \ldots \leq d_n$. Hence EDF schedules them in this order.

## Definition

A schedule $S$ is said to have an inversion if there are jobs $i$ and $j$ such that $S$ schedules $i$ before $j$, but $d_i > d_j$.

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A schedule $S$ is said to have an inversion if there are jobs $i$ and $j$ such that $S$ schedules $i$ before $j$, but $d_i > d_j$.

## Claim

*If a schedule $S$ has an inversion then there is an inversion between two* adjacently *scheduled jobs.*

Proof: exercise.

# Proof sketch of Optimality of EDP

- Let $S$ be an optimum schedule with smallest number of inversions.
- If $S$ has no inversions then this is same as EDF and we are done.
- Else $S$ has two adjacent jobs $i$ and $j$ with $d_i > d_j$.
- Swap positions of $i$ and $j$ to obtain a new schedule $S'$

**Claim**

*Maximum lateness of $S'$ is no more than that of $S$. And $S'$ has strictly fewer inversions than $S$.*

# Part IV

## Maximum Weight Subset of Elements: Cardinality and Beyond

# Picking k elements to maximize total weight

1. Given $n$ items each with non-negative weights/profits and integer $1 \leq k \leq n$.
2. Goal: pick $k$ elements to maximize total weight of items picked.

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
|---|---|---|---|---|---|---|
| *weight* | 3 | 2 | 1 | 4 | 3 | 2 |

$k = 2$:
$k = 3$:
$k = 4$:

# Greedy Template

```
N is the set of all elements X ← ∅
(* X will store all the elements that will be picked *)
while |X| < k and N is not empty do
    choose e_j ∈ N of maximum weight
    add e_j to X
    remove e_j from N
return the set X
```

**Remark:** One can rephrase algorithm simply as sorting elements in decreasing weight order and picking the top $k$ elements but the above template generalizes to other settings a bit more easily.

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**Remark:** One can rephrase algorithm simply as sorting elements in decreasing weight order and picking the top $k$ elements but the above template generalizes to other settings a bit more easily.

**Theorem**

*Greedy is optimal for picking $k$ elements of maximum weight.*

# A more interesting problem

1. Given $n$ items $N = \{e_1, e_2, \ldots, e_n\}$. Each item $e_i$ has a non-negative weight $w_i$.
2. Items partitioned into $h$ sets $N_1, N_2, \ldots, N_h$. Think of each item having one of $h$ colors.
3. Given integers $k_1, k_2, \ldots, k_h$ and another integer $k$
4. Goal: pick $k$ elements such that no more than $k_i$ from $N_i$ to maximize total weight of items picked.

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$, $e_7$ |
|---|---|---|---|---|---|---|
| *weight* | 3 | 2 | 1 | 4 | 3 | 2, 1 |

$N_1 = \{e_1, e_2, e_3\}$, $N_2 = \{e_4, e_5\}$, $N_3 = \{e_6, e_7\}$
$k = 5$, $k_1 = 2, k_2 = 2, k_3 = 2$

# Greedy Template

```
N is the set of all elements X ← ∅
(* X will store all the elements that will be picked *)
while N is not empty do
    N' = {e_i ∈ N | X ∪ {e_i} is feasible}
    If N' ← ∅ break
    choose e_j ∈ N' of maximum weight
    add e_j to X
    remove e_j from N
return the set X
```

# Greedy Template

$N$ is the set of all elements $X \leftarrow \emptyset$
(* $X$ will store all the elements that will be picked *)
**while** $N$ is not empty **do**
    $N' = \{e_i \in N \mid X \cup \{e_i\} \text{ is feasible}\}$
    If $N' \leftarrow \emptyset$ break
    choose $e_j \in N'$ of maximum weight
    add $e_j$ to $X$
    remove $e_j$ from $N$
**return** the set $X$

## Theorem

*Greedy is optimal for the problem on previous slide.*

Proof: exercise after class.

Special case of the general phenomenon of Greedy working for maximum weight indepedent set in a matroid. Beyond scope of

# Part V

# Interval Scheduling

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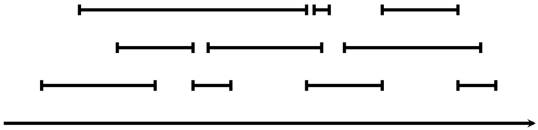
## Problem (Interval Scheduling)

**Input:** *A set of jobs with start and finish times to be scheduled on a resource (example: classes and class rooms).*
**Goal:** *Schedule as many jobs as possible*



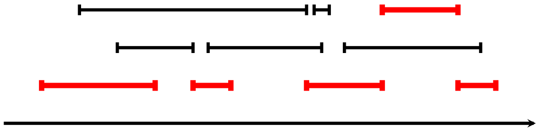
# Interval Scheduling

## Problem (Interval Scheduling)

**Input:** *A set of jobs with start and finish times to be scheduled on a resource (example: classes and class rooms).*

**Goal:** *Schedule as many jobs as possible*

1. *Two jobs with overlapping intervals cannot both be scheduled!*



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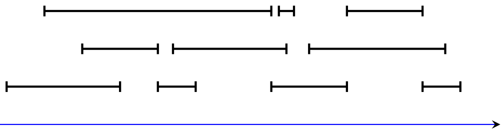
Main task: Decide the order in which to process requests in $\boldsymbol{R}$

ES SP FC EF

# Earliest Start Time

Process jobs in the order of their starting times, beginning with those that start earliest.



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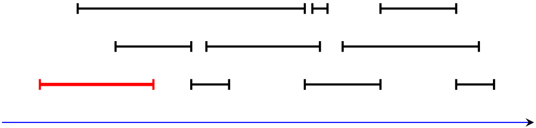

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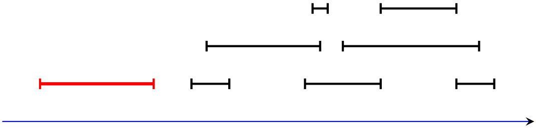

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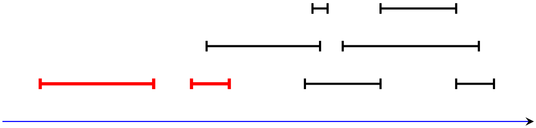

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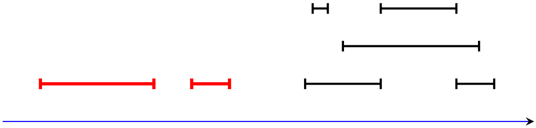

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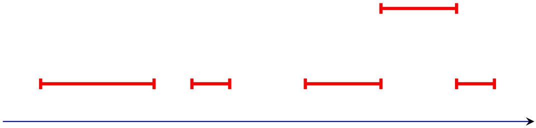

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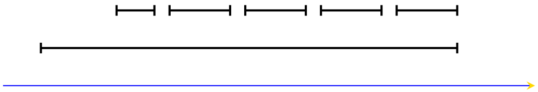

Figure: Counter example for earliest start time

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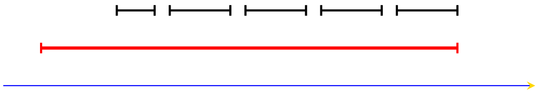

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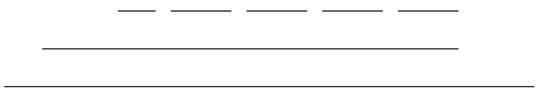

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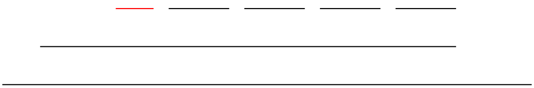

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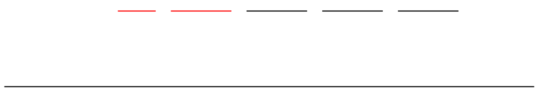

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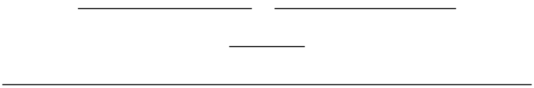

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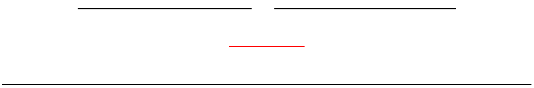

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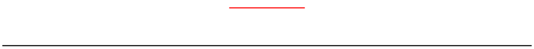

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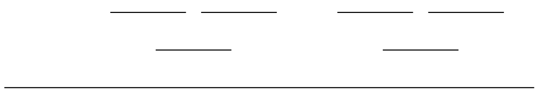

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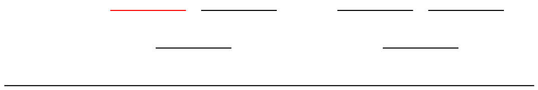

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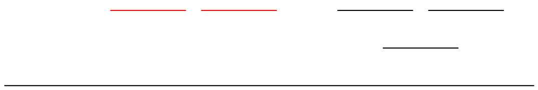

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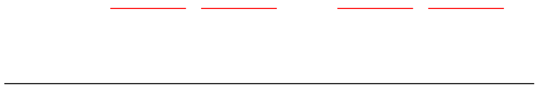

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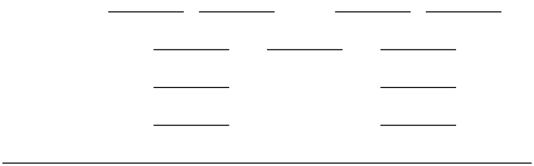

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Process jobs in that have the fewest "conflicts" first.

Figure: Counter example for fewest conflicts

Back Counter

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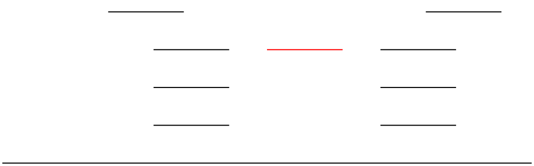

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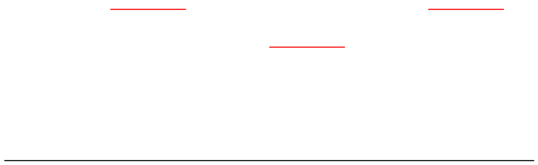

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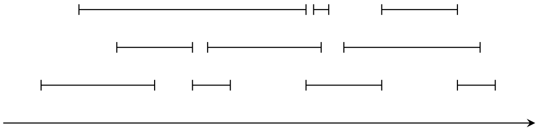
Back Counter

# Earliest Finish Time

Process jobs in the order of their finishing times, beginning with those that finish earliest.



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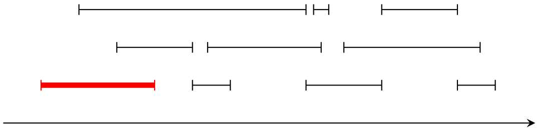

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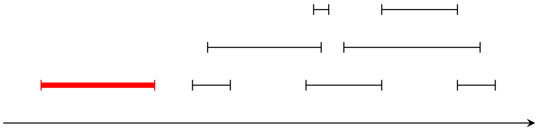

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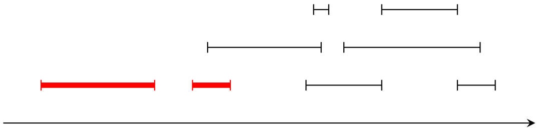

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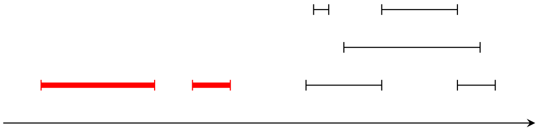

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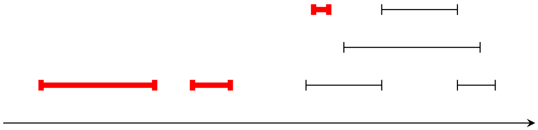

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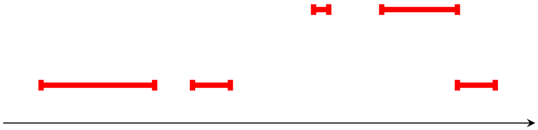

# Optimal Greedy Algorithm

```
R is the set of all requests
X ← ∅ (* X stores the jobs that will be scheduled *)
while R is not empty
    choose i ∈ R such that finishing time of i is smallest
    add i to X
    remove from R all requests that overlap with i
return X
```

**Theorem**

*The greedy algorithm that picks jobs in the order of their finishing times is optimal.*

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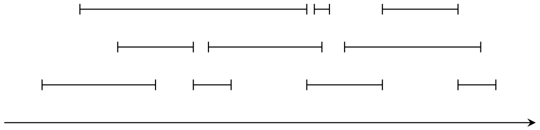

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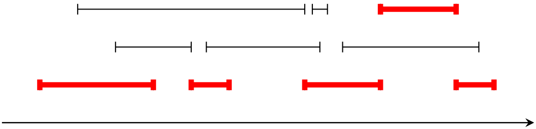

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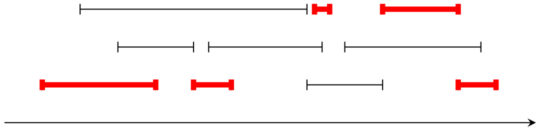

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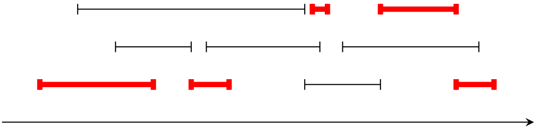

Instead we will show that $|O| = |X|$

# Proof of Optimality: Key Lemma

## Lemma

*Let $i_1$ be first interval picked by Greedy. There exists an optimum solution that contains $i_1$.*

## Proof.

Let $O$ be an *arbitrary* optimum solution. If $i_1 \in O$ we are done.

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**Claim:** If $i_1 \not\in O$ then there is exactly one interval $j_1 \in O$ that conflicts with $i_1$. (proof later)

1. Form a new set $O'$ by removing $j_1$ from $O$ and adding $i_1$, that is $O' = (O - \{j_1\}) \cup \{i_1\}$.
2. From claim, $O'$ is a *feasible* solution (no conflicts).
3. Since $|O'| = |O|$, $O'$ is also an optimum solution and it contains $i_1$. □

# Proof of Claim

## Claim

*If $i_1 \notin O$, there is exactly one interval $j_1 \in O$ that conflicts with $i_1$.*

## Proof.

1. If no $j \in O$ conflicts with $i_1$ then $O$ is not optimal!
2. Suppose $j_1, j_2 \in O$ such that $j_1 \neq j_2$ and both $j_1$ and $j_2$ conflict with $i_1$.
3. Since $i_1$ has earliest finish time, $j_1$ and $i_1$ overlap at $f(i_1)$.
4. For same reason $j_2$ also overlaps with $i_1$ at $f(i_1)$.
5. Implies that $j_1, j_2$ overlap at $f(i_1)$ but intervals in $O$ cannot overlap.

See figure in next slide. □

# Figure for proof of Claim

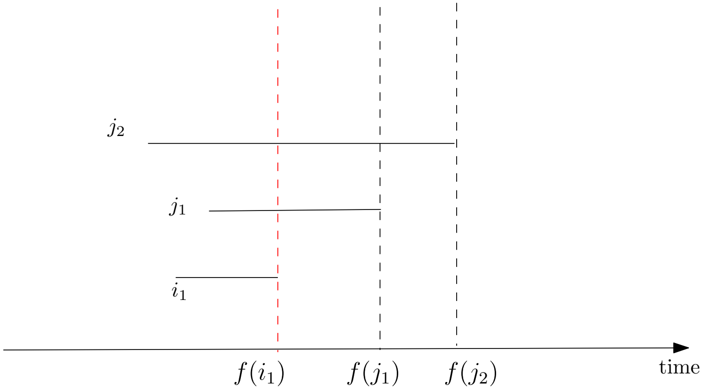


Figure: Since $i_1$ has the earliest finish time, any interval that conflicts with it does so at $f(i_1)$. This implies $j_1$ and $j_2$ conflict.

# Proof of Optimality of Earliest Finish Time First

## Proof by Induction on number of intervals.

**Base Case:** $n = 1$. Trivial since Greedy picks one interval.
**Induction Step:** Assume theorem holds for $i < n$.
Let $I$ be an instance with $n$ intervals
$I'$: $I$ with $i_1$ and all intervals that overlap with $i_1$ removed
$G(I), G(I')$: Solution produced by Greedy on $I$ and $I'$
From Lemma, there is an optimum solution $O$ to $I$ and $i_1 \in O$.
Let $O' = O - \{i_1\}$. $O'$ is a solution to $I'$.

$$\begin{aligned} |G(I)| &= 1 + |G(I')| \quad \text{(from Greedy description)} \\ &\geq 1 + |O'| \quad \text{(By induction, } G(I') \text{ is optimum for } I'\text{)} \\ &= |O| \end{aligned}$$

$\square$

# Implementation and Running Time

```
Initially R is the set of all requests
X ← ∅ (* X stores the jobs that will be scheduled *)
while R is not empty
    choose i ∈ R such that finishing time of i is least
    if i does not overlap with requests in X
        add i to X
    remove i from R
return the set X
```

- Presort all requests based on finishing time. $O(n \log n)$ time
- Now choosing least finishing time is $O(1)$
- Keep track of the finishing time of the last request added to $A$. Then check if starting time of $i$ later than that
- Thus, checking non-overlapping is $O(1)$
- Total time $O(n \log n + n) = O(n \log n)$

# Comments

1. Interesting Exercise: smallest interval first picks at least half the optimum number of intervals.
2. All requests need not be known at the beginning. Such *online* algorithms are a subject of research

# Weighted Interval Scheduling

Suppose we are given $n$ jobs. Each job $i$ has a start time $s_i$, a finish time $f_i$, and a weight $w_i$. We would like to find a set $S$ of compatible jobs whose total weight is maximized. Which of the following greedy algorithms finds the optimum schedule?

- **(A)** Earliest start time first.
- **(B)** Earliest finish time fist.
- **(C)** Highest weight first.
- **(D)** None of the above.
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**(D)** None of the above.

**(E)** IDK.

Weighted problem can be solved via dynamic prog. See notes.

# Greedy Analysis: Overview

1. Greedy's first step leads to an optimum solution. Show that there is an optimum solution leading from the first step of Greedy and then use induction. Example, Interval Scheduling.
2. Greedy algorithm stays ahead. Show that after each step the solution of the greedy algorithm is at least as good as the solution of any other algorithm. Example, Interval scheduling.
3. Structural property of solution. Observe some structural bound of every solution to the problem, and show that greedy algorithm achieves this bound. Example, Interval Partitioning (see Kleinberg-Tardos book).
4. Exchange argument. Gradually transform any optimal solution to the one produced by the greedy algorithm, without hurting its optimality. Example, Minimizing lateness.

# Takeaway Points

1. Greedy algorithms come naturally but often are incorrect. A proof of correctness is an absolute necessity.
2. *Exchange* arguments are often the key proof ingredient. Focus on why the first step of the algorithm is correct: need to show that there is an optimum/correct solution with the first step of the algorithm.
3. Thinking about correctness is also a good way to figure out which of the many greedy strategies is likely to work.