

# Turing Machines

# “Most General” computer?

- DFAs are simple model of computation.
  - Accept only the regular languages.
- Is there a kind of computer that can accept *any* language, or compute *any* function?
- Recall counting argument:
  - $\{L \mid L \subseteq \{0,1\}^*\}$  (just the set languages)
    - (a) countably infinite
    - (b) uncountably infinite
  - $\{P : P \text{ is a finite length computer program}\}$  is
    - (a) countably infinite
    - (b) uncountably infinite

# *Most General Computer*

- If not all functions are computable, *which are?*
- Is there a “most general” model of computer?
- What languages can they recognize?

# *David Hilbert*

- Early 1900s – crisis in math foundations
  - attempts to formalize resulted in paradoxes, etc.
- 1920, Hilbert’s Program:
  - “mechanize” mathematics
- Finite axioms, inference rules
  - turn crank, determine truth
  - needed: axioms *consistent & complete*



# *Kurt Gödel*

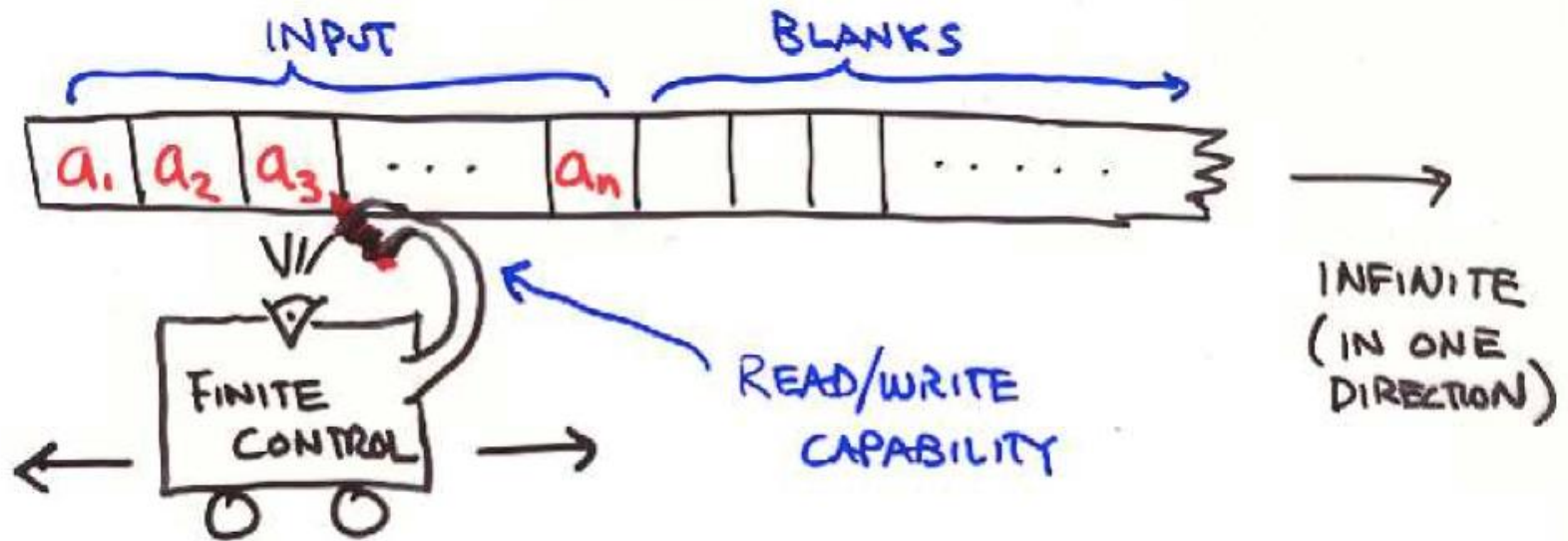
- German logician, at age 25 (1931) proved:  
“There are true statements that can’t be proved”  
(i.e., “no” to Hilbert)
- Shook the foundations of
  - mathematics
  - philosophy
  - science
  - everything



# Alan Turing

- British mathematician
  - cryptanalysis during WWII
  - arguably, father of AI, Theory
  - several books, movies
- Defined “*computer*”, “*program*”  
and (1936) at age 23 provided foundations for investigating fundamental question of what is computable, what is not computable.





- DFA with (infinite) tape.
- One move: read, **write**, move, change state.

# *High-level Goals*

- **Church-Turing thesis:** TMs are the most general computing devices. So far no counter example
- Every TM can be represented as a string. Think of TM as a program but in a very low-level language.
- Existence of **Universal Turing Machine** which is the model/inspiration for stored program computing. UTM can simulate any TM
- Implications for what can be computed and what cannot be computed



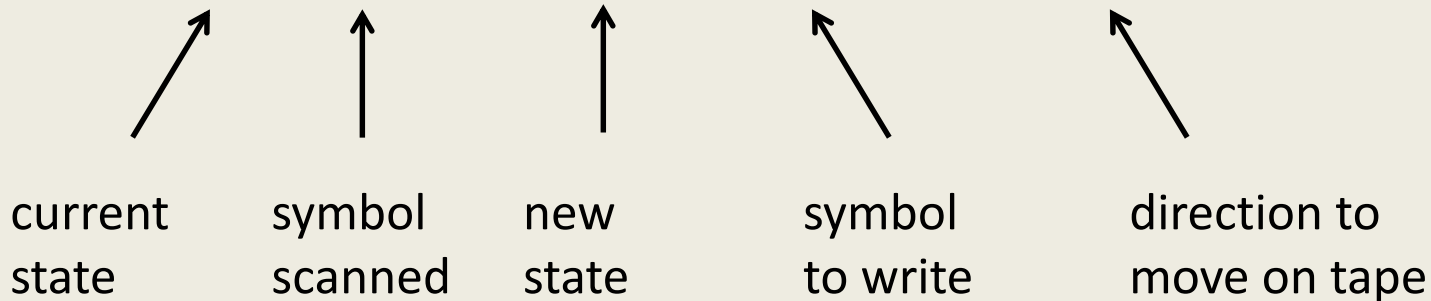
## *Formal Definition*

$M = (Q, \Sigma, \Gamma, \delta, q_0, B, q_{accept}, q_{reject})$ , where:

- $Q$  is a finite set of states
- $\Sigma$  is a finite input alphabet
- $\delta$  as defined on next page
- $\Gamma$  is a finite tape alphabet. ( $\Sigma$  a subset of  $\Gamma$ )
- $q_0$  is the initial state (in  $Q$ )
- $B$  in  $\Gamma - \Sigma$  is the blank symbol
- $q_{accept}, q_{reject}$  are unique accept, reject states in  $Q$

# Transition Function

$$\delta: Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$$



$$\delta(q, a) = (p, b, L)$$

from state  $q$ , on reading  $a$ :

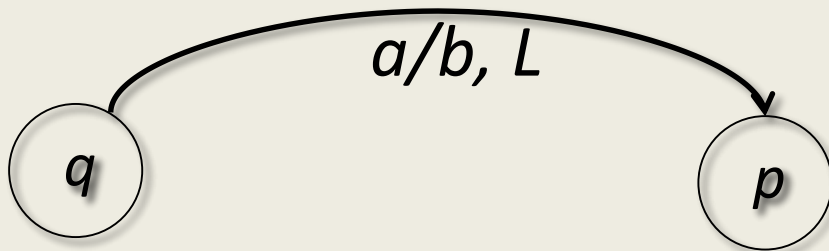
go to state  $p$

write  $b$

move head **Left**

# Graphical Representation

$$\delta(q, a) = (p, b, L)$$

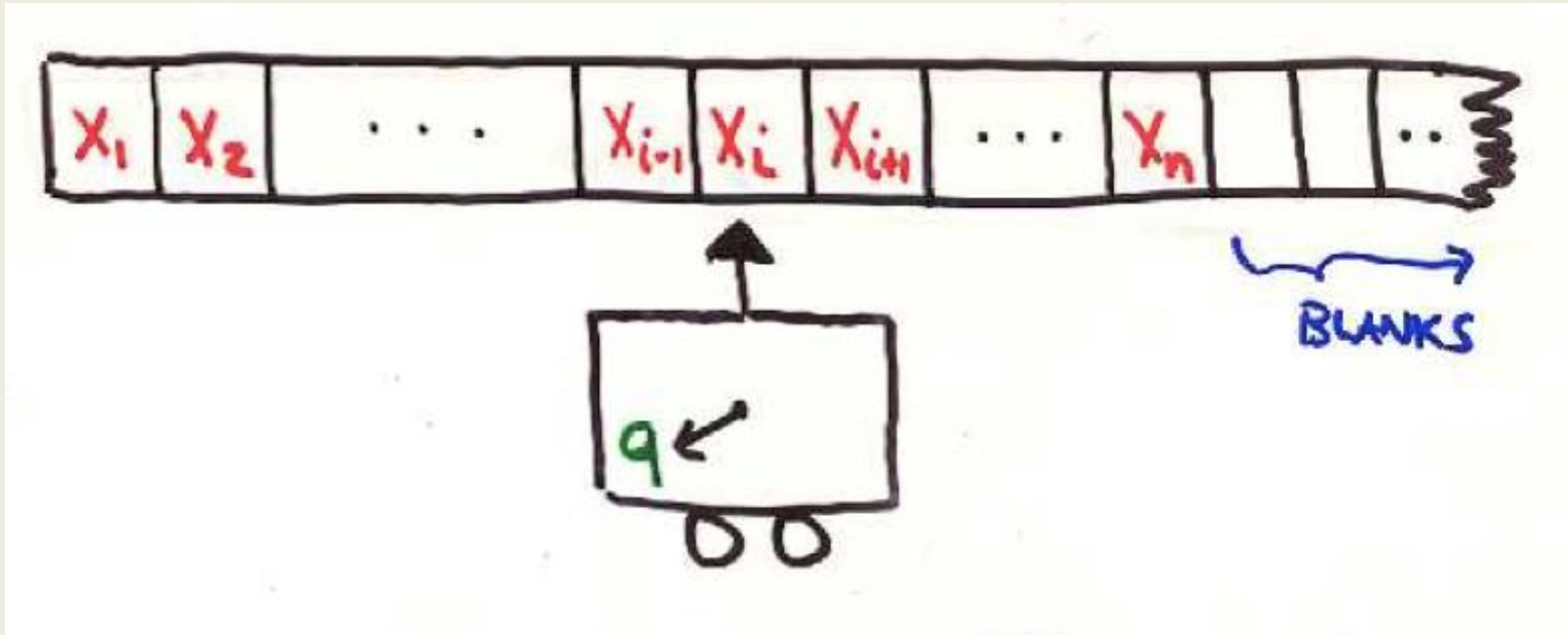


Note: we allow  $\delta(q, a)$  to be undefined for some choices of  $q, a$  (in which case,  $M$  “crashes”)

## *ID: Instantaneous Description*

- Contains all necessary information to capture “state of the computation”
- Includes
  - state  $q$  of  $M$
  - location of read/write head
  - contents of tape from left edge to rightmost nonblank (or to head, whichever is rightmost)

# *ID: Instantaneous Description*



ID:  $X_1 X_2 \dots X_{i-1} q X_i X_{i+1} \dots X_n$

( $q$  in  $Q$ ,  $X_i$  in  $\Gamma$ )

## Relation “ $\rightarrow$ ” on IDs

If  $\delta(q, X_i) = (p, Y, L)$ , then

$$\underbrace{X_1 X_2 \dots X_{i-1} q X_i X_{i+1} \dots X_n}_{\text{current ID}} \rightarrow \underbrace{X_1 X_2 \dots X_{i-2} p X_{i-1} Y X_{i+1}}_{\text{next ID}}$$

If  $\delta(q, X_i)$  is undefined, then there is no next ID

If  $M$  tries to move off left edge, there is no next ID  
(in both cases, the machine “crashes”)

## *Capturing many moves...*

Define  $\rightarrow^*$  as the reflexive, transitive closure of  $\rightarrow$

Thus,  $ID_1 \rightarrow^* ID_2$  iff  $M$ , when run from  $ID_1$ , necessarily reaches  $ID_2$  after some finite number of moves.

Initial ID:  $q_0 w$  (more often, assume ...  $\$q_0 w$ )

Accepting ID:  $\alpha_1 q_{accept} \alpha_2$  for any  $\alpha_1, \alpha_2$  in  $\Gamma^*$

(reaches the accepting state with any random junk left on the tape)

# Definition of Acceptance

$M$  accepts  $w$  iff for some  $\alpha_1, \alpha_2$  in  $\Gamma^*$ ,

$$q_0 w \rightarrow^* \alpha_1 q_{\text{accept}} \alpha_2$$

$M$  accepts if at *any* time it enters the accept state

Regardless of whether or not

it has scanned all of the input

it has moved back and forth many times

it has completely erased or replaced  $w$  on the tape

$$L(M) = \{w \mid M \text{ accepts } w\}$$



# *Non-accepting computation*

$M$  doesn't accept  $w$  if any of the following occur:

- $M$  enters  $q_{reject}$
- $M$  moves off left edge of tape
- $M$  has no applicable next transition
- $M$  continues computing forever

If  $M$  accepts – we can tell: it enters  $q_{accept}$

If  $M$  doesn't accept – we may not be able to tell

(c.f. “Halting problem” – later)

# “Recursive” vs “Recursively Enumerable”

- *Recursively Enumerable (r.e.) Languages:*  
=  $\{L \mid \text{there is a TM } M \text{ such that } L(M) = L\}$
- *Recursive Languages* (also called “**decidable**”)  
=  $\{L \mid \text{there is a TM } M \text{ that halts for all } w \text{ in } \Sigma^* \text{ and such that } L(M) = L\}$

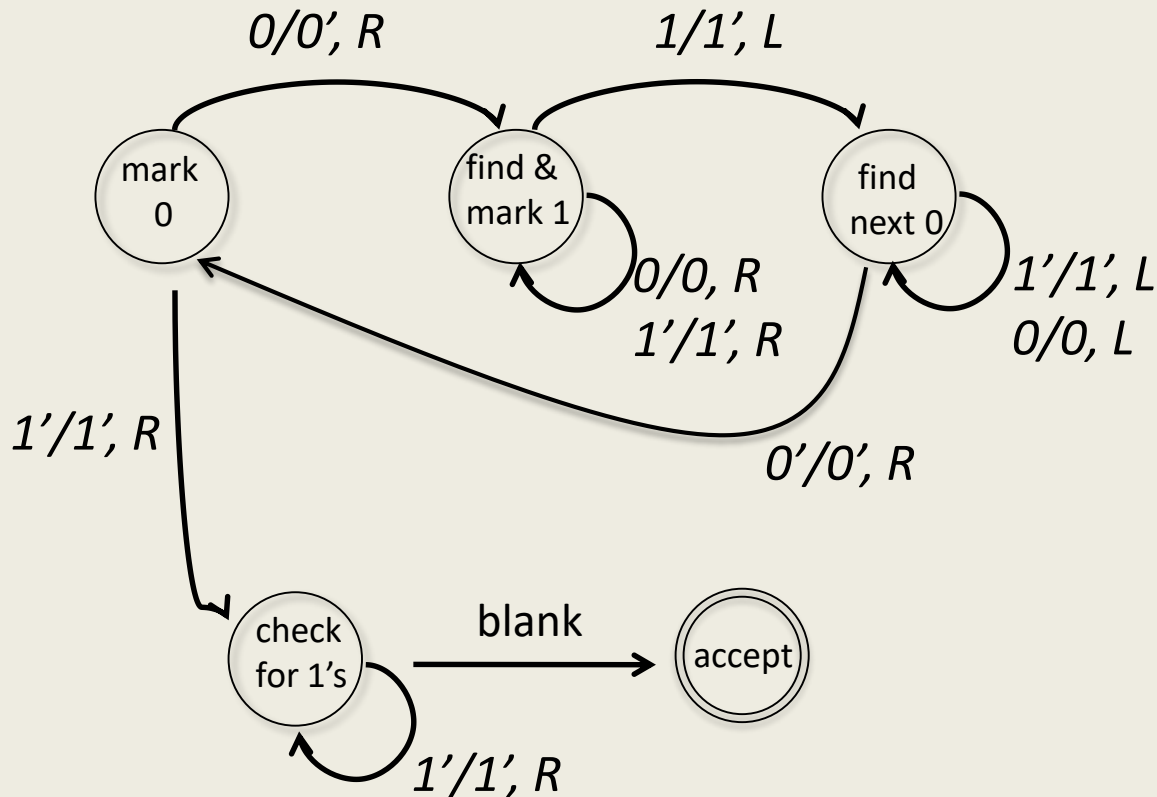
Recursive languages: nice; run  $M$  on  $w$  and it will eventually enter either  $q_{\text{accept}}$  or  $q_{\text{reject}}$

r.e. languages: not so nice; can know if  $w$  in  $L$ , but not necessarily if  $w$  is not in  $L$ .

# *Fundamental Questions*

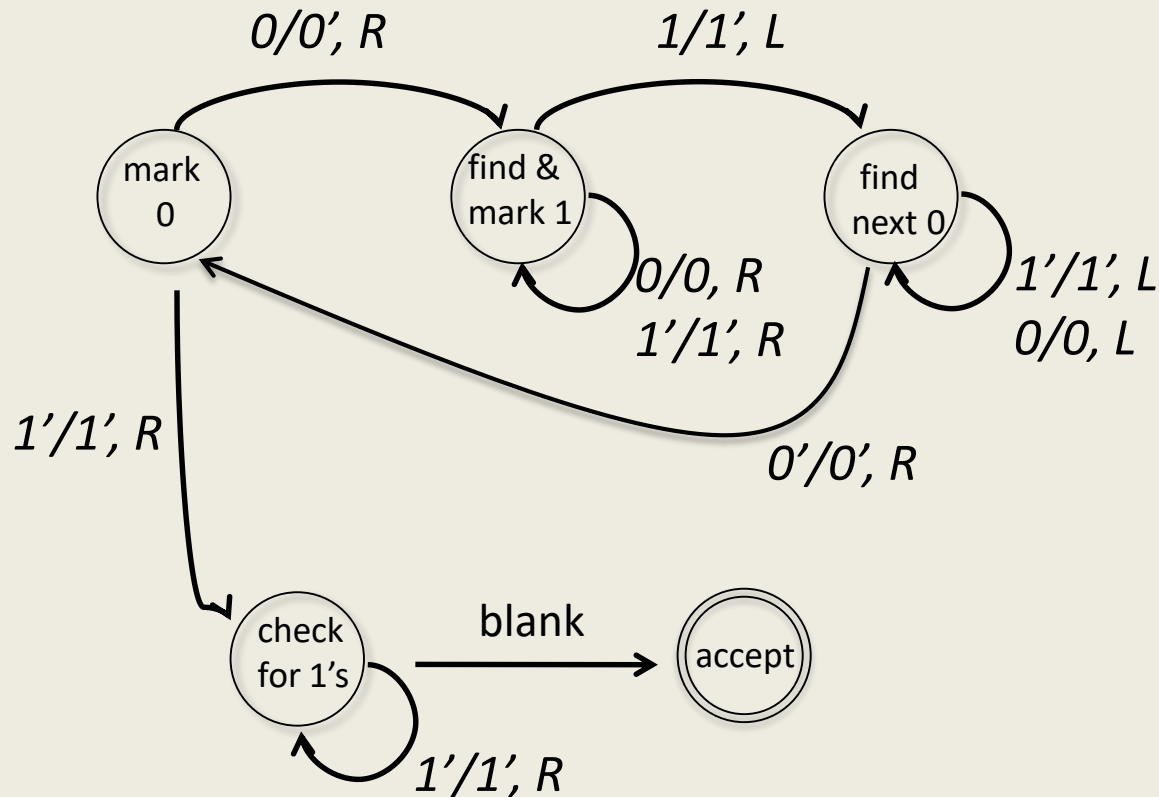
- Which languages are r.e.?
- Which are recursive?
- What is the difference?
- What properties make a language decidable?

# Machine accepting $\{0^n 1^n \mid n \geq 1\}$



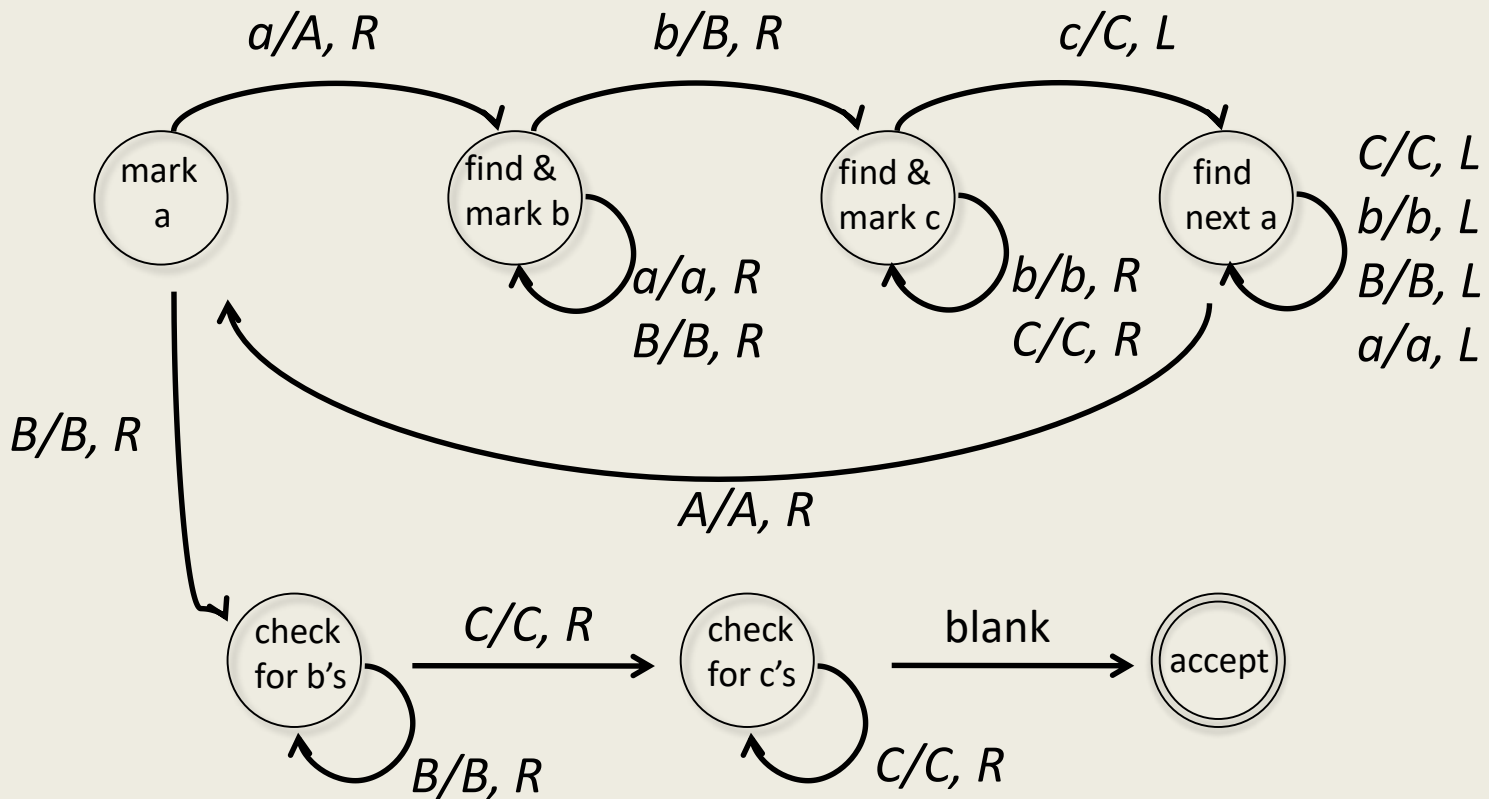
(This technique is known as “checking off symbols”)

# Machine accepting $\{0^n 1^n \mid n \geq 1\}$



(This technique is known as “checking off symbols”)

# Machine accepting $\{a^n b^n c^n \mid n \geq 1\}$



(This technique is known as "checking off symbols")

# *Machine to add two $n$ -bit numbers*

*(“high-level” description)*

- Assume input is  $a_1a_2\dots a_n\#b_1b_2\dots b_n$
- Pre-process phase
  - sweep right, replacing 0 with 0' and 1 with 1'
- Main loop:
  - erase last bit  $b_i$ , and remember it
  - move left to corresponding bit  $a_i$
  - add the bits, plus carry, overwrite  $a_i$  with answer
  - remember carry, move right to next (last) bit  $b_{i-1}$

Program Trace (some missing steps)

\$10011#11001

\$1'0'0'1'1'#1'1'0'0'1'

\$1'0'0'1'1'#1'1'0'0'1  $b = 1$   
 $c = 0$

\$1'0'0'1'1'#1'1'0'0'0

\$1'0'0'1'1'#1'1'0'0'

\$1'0'0'1'1'#1'1'0'0'

\$1'0'0'1'1'#1'1'0'0'

\$1'0'0'1'1'#1'1'0'0'

\$1'0'0'1'1'#1'1'0'0'

\$1'0'0'1'0#1'1'0'0'  $c = 1$

\$1'0'0'1'0#1'1'0'0'

\$1'0'0'1'0#1'1'0'0'

\$1'0'0'1'0#1'1'0'0'

\$1'0'0'1'0#1'1'0'0'

\$1'0'0'1'0#1'1'0'0'

\$1'0'0'1'0#1'1'0'  $b = 0$   
 $c = 1$

\$1'0'0'1'0#1'1'0'

\$1'0'0'1'0#1'1'0'

\$1'0'0'10#1'1'0'

\$1'0'0'00#1'1'0'  $c = 1$

\$1'0'0'00#1'1'0' etc



# Computing Functions with TMs

- number  $n$  represented in unary by  $0^n$   
*(well, really  $0^{n+1}$  so we can represent 0...)*
- $M(n)$  definition: “output of  $M$  on input  $n$ ”:  
IF  $q_0 0^n \rightarrow^* q_{halt} 0^m$  then  $M(n) = m$ .
- Every TM  $M$  computes some function  
 $f_M : \mathbb{N} \rightarrow \mathbb{N} \cup \{\text{undefined}\}$ .
- Functions with multiple inputs and outputs:  
 $M(x,y,z) = (r,s)$  means  $q_0 0^x \# 0^y \# 0^z \rightarrow^* q_{halt} 0^r \# 0^s$

# *“Easily” Computed Functions*

- addition
- multiplication
- subtraction ( $\max \{0, x-y\}$ )
- any function you have an “algorithm” for...

# Computable Functions

- A (partial) function  $f: \mathbb{N} \rightarrow \mathbb{N} \cup \{\text{undefined}\}$  is said to be *computable* iff for some TM  $M$ ,  
for all  $x$  in  $\mathbb{N}$ ,  $f(x) = M(x)$  when  $f(x)$  is defined  
“ $f$  is a (partial) recursive function”
- A function  $f: \mathbb{N} \rightarrow \mathbb{N}$  is a *total recursive function* iff for all  $x$  in  $\mathbb{N}$ ,  $f(x) = M(x)$  for every  $x$ .
- If  $M$  computes a partial recursive function, it may not halt on some inputs. If  $M$  computes a total recursive function, it must halt for all inputs.

## *Why only $f: \mathbb{N} \rightarrow \mathbb{N}$ ?*

Q: What about negatives, rationals, reals?

A: We can encode anything as a natural number.

- -5: 100000
- $p/q$ :  $0^p 1 1 0^q$ ,  $(p/q) + (a/b)$  as  $0^{pb+aq} 1 1 0^{bq}$
- reals to given precision, or symbolically
- **Intuition: ANY function can be coded as a function from  $\mathbb{N}$  to  $\mathbb{N}$**

## *Some TM programming tricks*

- checking off symbols
- shifting over
- using finite control memory
- subroutine calls

# *“Extensions” of TMs*

- 2-way infinite tape
- multiple tracks
- multiple tapes
- multi-dimensional TMs
- nondeterministic TMs
- --- bells & whistles

Goal:

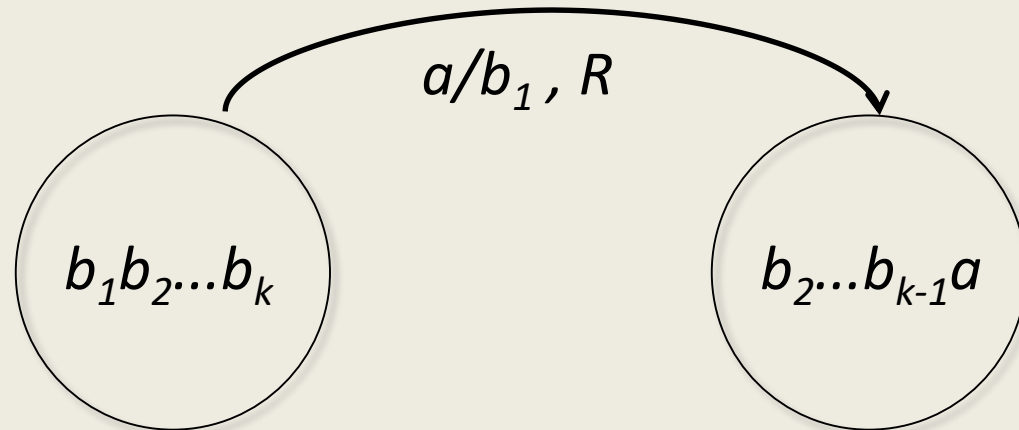
Convince you of the power of the basic model

## *Checking off symbols*

- Use additional tape symbols to represent a “checked-off” character.
- E.g., for each symbol  $a$  in  $\Gamma$ , also include “ $a_{\checkmark}$ ” to represent a marked  $a$ .
- We essentially did this using  $A$  for checked  $a$ , or  $0'$  for checked  $0$ , in our previous examples.

# Shifting over

- Sometimes need extra cells
- Can shift-over by any number of cells
  - Shift-by-k: Use states to remember previous  $|\Gamma|^k$  symbols:



plus states to begin and end the process



## *Implication of Shifting*

- Can assume without loss of generality that input is in the form  $\$w$  where  $\$$  is a special symbol at start of input and not used anywhere else
- With above assumption can avoid the issue of crashing because of moving off the left of the tape

## *Using finite control*

- just like DFAs
- can use tuples to store different types of info
- E.g.,  $\{a^n b^n c^n \mid n = 1 \pmod{4} \text{ and } n = 2 \pmod{7}\}$

States:  $(q, i, j)$  where:

$q$ : state from TM for  $\{a^n b^n c^n \mid n \geq 1\}$

$i$ : counter mod 4

$j$ : counter mod 7

etc.

In general, can store *any finite information* in states

# *Subroutine calls*

- soon

## *“Extensions” of TMs: 2-way infinite tape*

.	.	-5	-4	-3	-2	-1	0	1	2	3	4	5	.	.
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Simulate with 1-way infinite tape...

0	1	-1	2	-2	3	-3	4	-4	5	-5	6	.	.	.
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Must modify transitions appropriately

- remember in finite control if negative or positive
- if positive,  $R \rightarrow RR$ ;  $L \rightarrow LL$
- if negative,  $R \rightarrow LL$ ;  $L \rightarrow RR$
- must mark left edge & deal with 0 cell differently

# Extension: multiple tracks

4 tracks

0	1	1	0						
\$	1	0	0	1					
a	b	b	c	a	a	a			
		2							

infinite tape →

M can address any particular track in the cell it is scanning

Can simulate 4 tracks with a single track machine, using extra “stacked” characters:

single character →

0	1	1	0						
\$	1	0	0	1					
a	b	b	c	a	a	a			
		2							

# Multiple tracks

4 tracks

0	1	1	0						
\$	1	0	0	1					
a	b	b	c	a	a	a			
		2							

infinite tape →

$$M: \delta(q, -, 0, -, -) = (p, -, -, -, 1, R)$$

“If in state  $q$  reading 0 on second track, then go to state  $p$ , write 1 on fourth track, and move right”

Then in  $M'$   $\delta(q, \begin{matrix} x \\ 0 \\ y \\ z \end{matrix} ) = (p, \begin{matrix} x \\ 0 \\ y \\ 1 \end{matrix} , R)$

for every  $x, y, z$  in  $\Gamma$


# *Extension: multiple tapes*

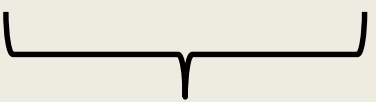
## *k*-tape TM

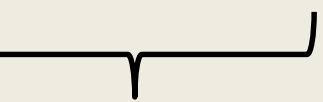
- $k$  different (2-way infinite) tapes
- $k$  different independently controllable heads
- input initially on tape 1; tapes 2, 3, ...,  $k$ , blank.
- single move:
  - read symbols under all heads
  - print (possibly different) symbols under heads
  - move all heads (possibly different directions)
  - go to new state

# *k*-tape TM transition function

$$\delta(q, a_1, a_2, \dots, a_k) = (p, b_1, b_2, \dots, b_k, D_1, D_2, \dots, D_k)$$

  
Symbols scanned on  
the *k* different tapes

  
Symbols to be written  
on the *k* different tapes

  
Directions to be moved  
( $D_i$  is one of L, R, S)

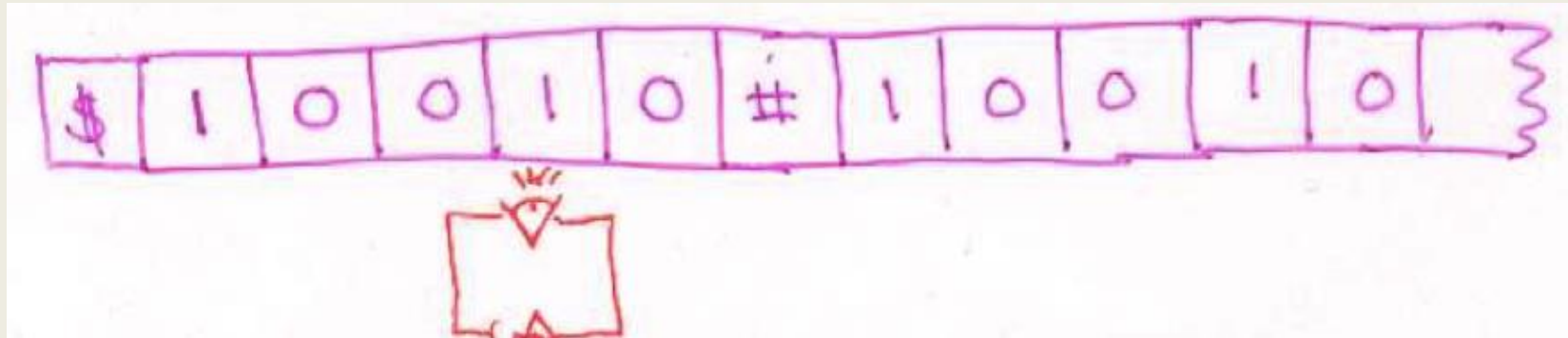
## Utility of multiple tapes

makes programming a whole lot easier

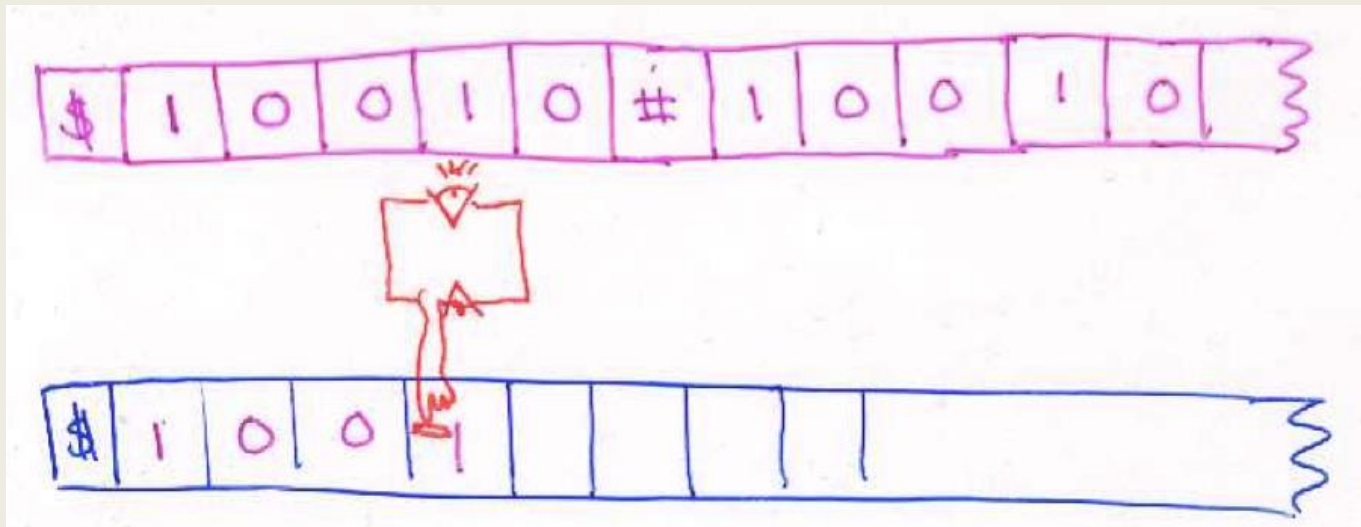
\$	1	0	0	1	0	#	1	0	0	1	0			
----	---	---	---	---	---	---	---	---	---	---	---	--	--	--

*is input string of form  $w\#w$  ?*





$\Omega(n^2)$  steps provably required

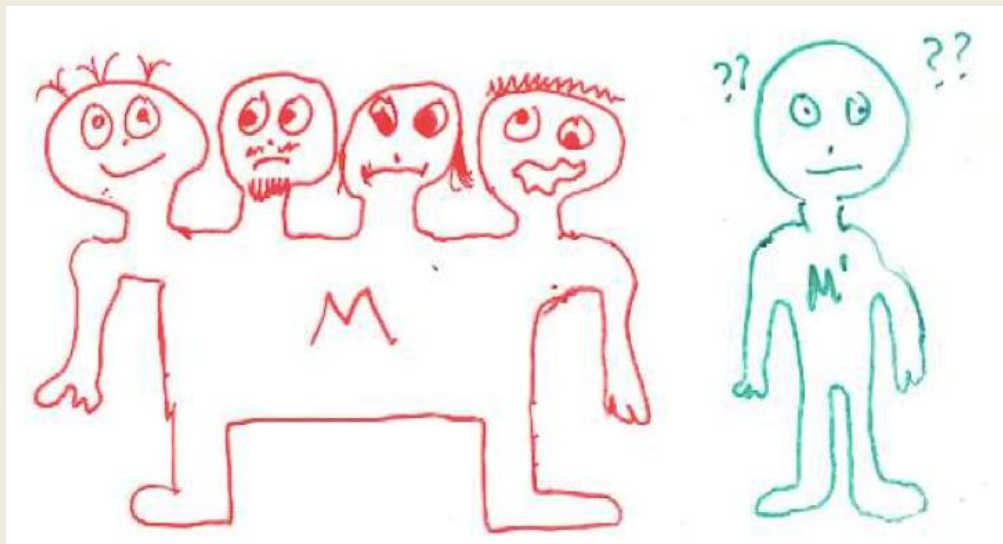


$\approx 3n/2$  steps easily programmed

# *Can't compute more with $k$ tapes*

*Theorem: If  $L$  is accepted by a  $k$ -tape TM  $M$ ,  
then  $L$  is accepted by some 1-tape TM  $M'$ .*

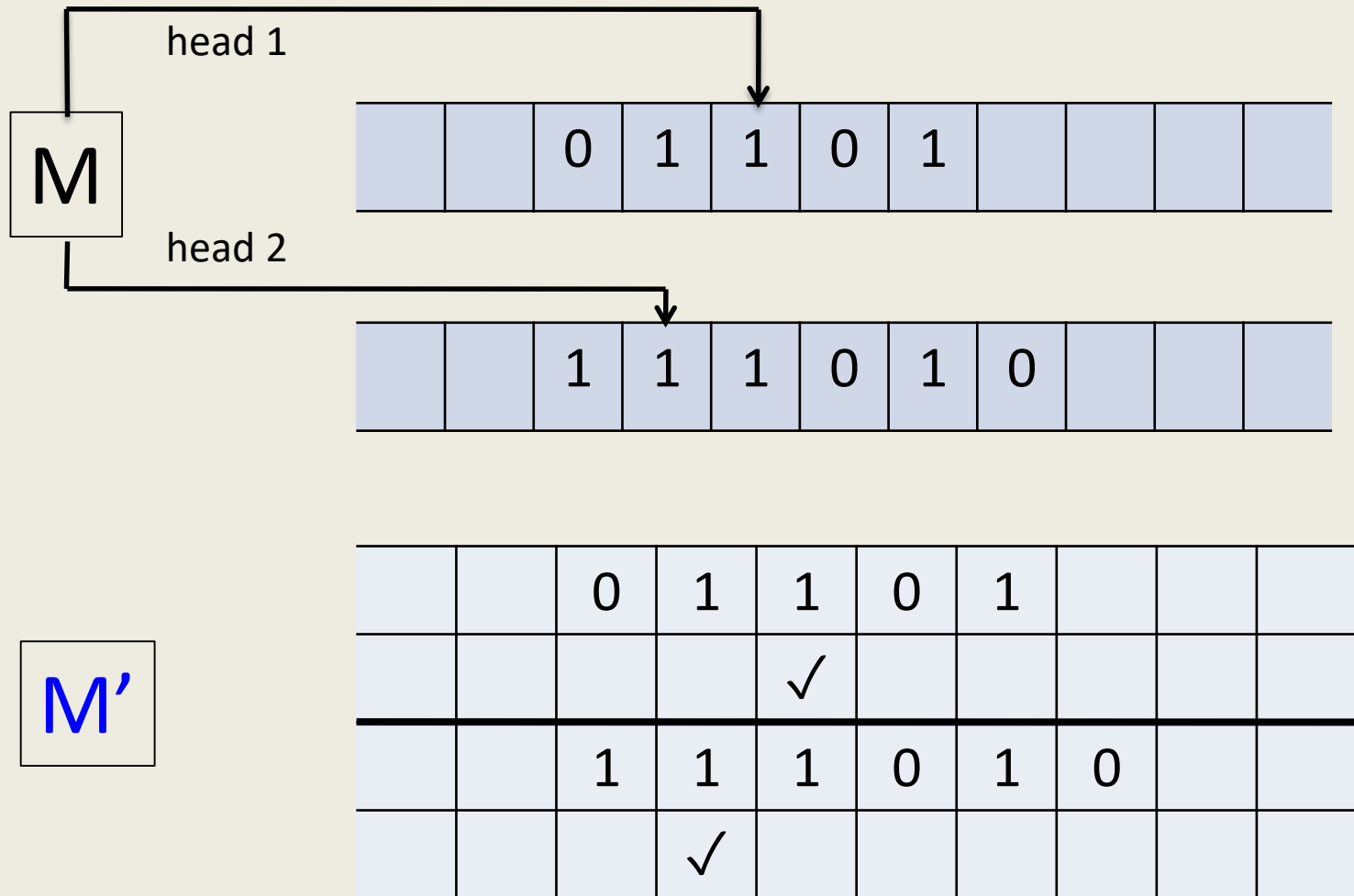
*Intuition:  $M'$  uses  $2k$  tracks to simulate  $M$*



*BUT....  
 $M$  has  $k$  heads!*

*How can  $M'$  be in  
 $k$  places at once?*

# Snapshot of simulation ( $k = 2$ )



Track  $2i-1$  holds tape  $i$ . Track  $2i$  holds position of head  $i$

*To make a move,  $M'$  does:*

**Phase 1:** Sweep from leftmost edge to rightmost “√” on any track, noting symbols √'ed, and what track they are on. Save this info in finite control.

Now,  $M'$  knows what move of  $M$  to make

**Phase 2:** Sweep from right to left edge implementing the move of  $M$

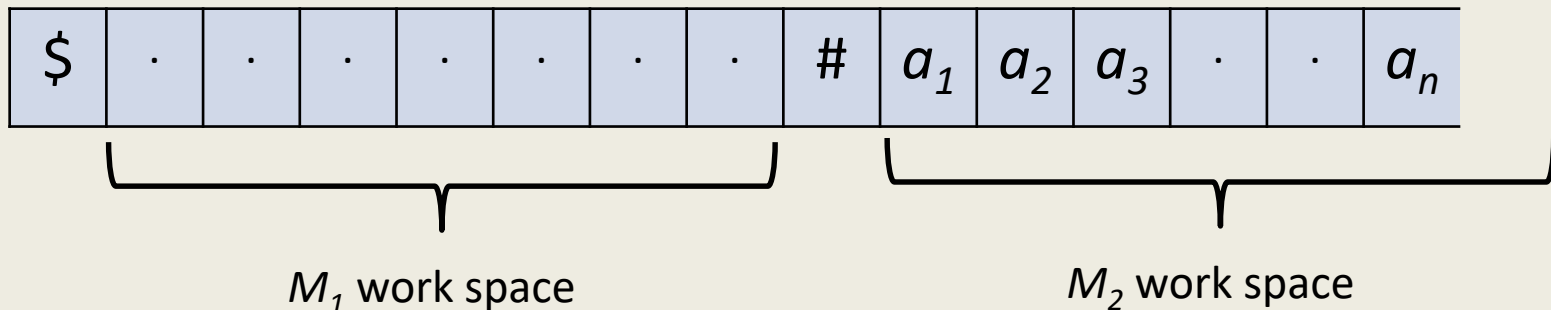
Thus, each move of  $M$  requires  $M'$  to do a complete sweep across, and back.

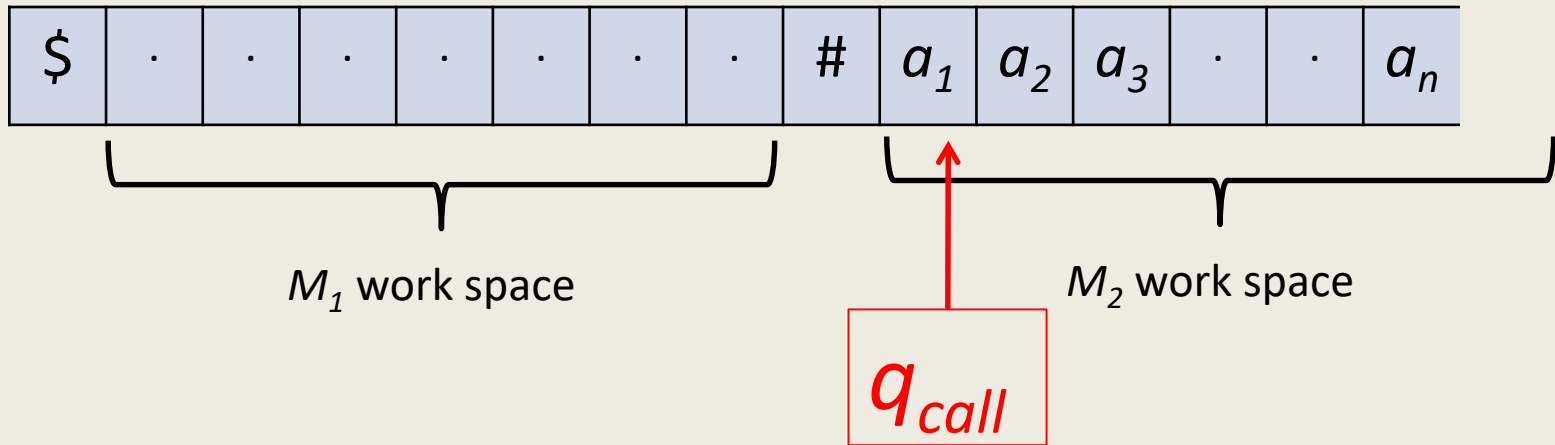
Not hard to show that if  $M$  takes  $t$  steps to complete its computation, then  $M'$  takes  $O(t^2)$  steps.

# Subroutine calls

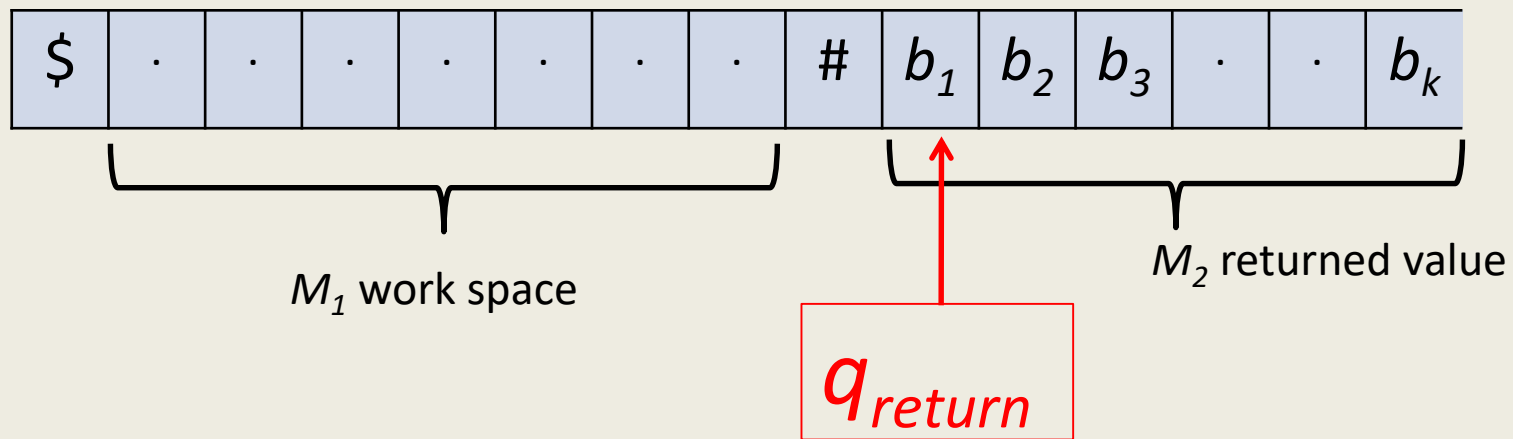
Mechanism for  $M_1$  to “call”  $M_2$  on an argument

- Rename states so that  $M_1$  and  $M_2$  have no common states *except*  $q_{call}$  and  $q_{return}$
- Goal:  $M_1$  calls from state  $q_{call}$  returns to  $q_{return}$
- Rename init. state of  $M_2$  as  $q_{call}$ , halt state  $q_{return}$
- $M_1$  sets up argument  $a_1a_2\dots a_n$  for  $M_2$  :

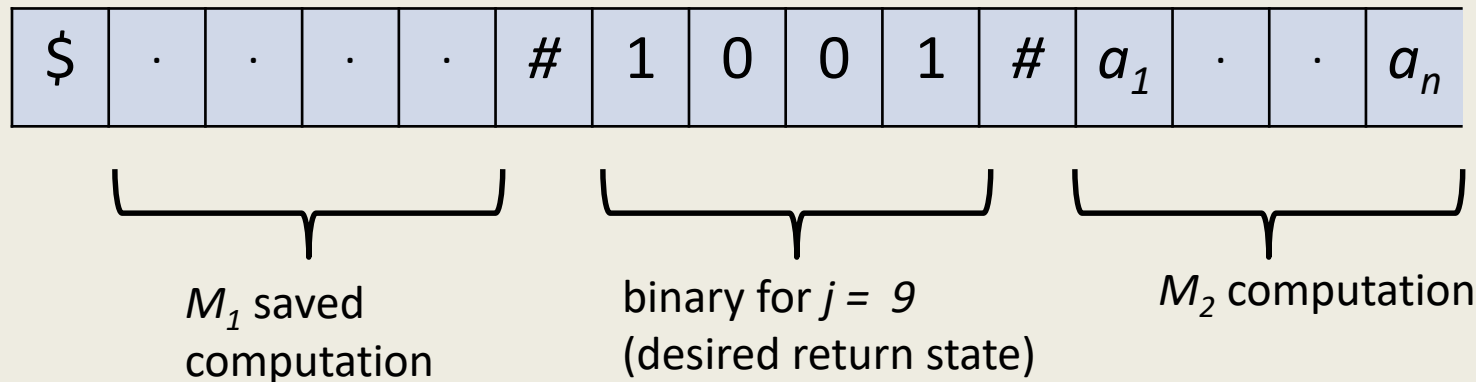




- $M_2$  runs, and when done:



- Can be more elaborate, and return to specified state  $q_j$



- $q_{return}$  now goes to special sequence of states designed to read binary 1011 (or whatever) and then transition to state 9 (or whatever)
  - This can actually be done with a DFA