CS 374: Algorithms & Models of Computation

Chandra Chekuri

University of Illinois, Urbana-Champaign

Spring 2017

NFAs continued, Closure Properties of Regular Languages

Lecture 5

January 31, 2017

Regular Languages, DFAs, NFAs

Theorem

Languages accepted by DFAs, NFAs, and regular expressions are the same.

Regular Languages, DFAs, NFAs

Theorem

Languages accepted by DFAs, NFAs, and regular expressions are the same.

- DFAs are special cases of NFAs (trivial)
- NFAs accept regular expressions (we saw already)
- DFAs accept languages accepted by NFAs (today)
- Regular expressions for languages accepted by DFAs (later in the course)

Part I

Equivalence of NFAs and DFAs

Equivalence of NFAs and DFAs

Theorem

For every NFA N *there is a DFA* M *such that* $L(M) = L(N)$.

Formal Tuple Notation for NFA

Definition

A non-deterministic finite automata (NFA) $N = (Q, \Sigma, \delta, s, A)$ is a five tuple where

- Q is a finite set whose elements are called states,
- Σ is a finite set called the input alphabet,
- $\delta : Q \times \Sigma \cup \{\epsilon\} \rightarrow \mathcal{P}(Q)$ is the transition function (here $\mathcal{P}(Q)$ is the power set of Q),
- $s \in Q$ is the start state,
- $A \subseteq Q$ is the set of accepting/final states.

$\delta(q, a)$ for $a \in \Sigma \cup \{\epsilon\}$ is a susbet of Q — a set of states.

Extending the transition function to strings

Definition

For NFA $N = (Q, \Sigma, \delta, s, A)$ and $q \in Q$ the $\epsilon\text{reach}(q)$ is the set of all states that q can reach using only ϵ-transitions.

Definition

Inductive definition of $\delta^* : Q \times \Sigma^* \to \mathcal{P}(Q)$:

- if $w = \epsilon$, $\delta^*(q, w) = \epsilon\text{reach}(q)$
- if $w = a$ where $a \in \Sigma$
  $\delta^*(q, a) = \cup_{p \in \epsilon\text{reach}(q)}(\cup_{r \in \delta(p,a)} \epsilon\text{reach}(r))$
- if $w = xa$,
  $\delta^*(q, w) = \cup_{p \in \delta^*(q,x)}(\cup_{r \in \delta(p,a)} \epsilon\text{reach}(r))$

Formal definition of language accepted by N

Definition

A string w is accepted by NFA N if $\delta_N^*(s, w) \cap A \neq \emptyset$.

Definition

The language $L(N)$ accepted by a NFA $N = (Q, \Sigma, \delta, s, A)$ is

$$\{w \in \Sigma^* \mid \delta^*(s, w) \cap A \neq \emptyset\}.$$

Simulating an NFA by a DFA

- Think of a program with fixed memory that needs to simulate NFA N on input w.
- What does it need to store after seeing a prefix x of w?

Simulating an NFA by a DFA

- Think of a program with fixed memory that needs to simulate NFA N on input w.
- What does it need to store after seeing a prefix x of w?
- It needs to know at least $\delta^*(s, x)$, the set of states that N could be in after reading x
- Is it sufficient?

Simulating an NFA by a DFA

- Think of a program with fixed memory that needs to simulate NFA N on input w.
- What does it need to store after seeing a prefix x of w?
- It needs to know at least $\delta^*(s, x)$, the set of states that N could be in after reading x
- Is it sufficient? Yes, if it can compute $\delta^*(s, xa)$ after seeing another symbol a in the input.
- When should the program accept a string w?

Simulating an NFA by a DFA

- Think of a program with fixed memory that needs to simulate NFA N on input w.
- What does it need to store after seeing a prefix x of w?
- It needs to know at least $\delta^*(s, x)$, the set of states that N could be in after reading x
- Is it sufficient? Yes, if it can compute $\delta^*(s, xa)$ after seeing another symbol a in the input.
- When should the program accept a string w? If $\delta^*(s, w) \cap A \neq \emptyset$.

Key Observation: A DFA M that simulates N should keep in its memory/state the set of states of N

Thus the state space of the DFA should be $\mathcal{P}(Q)$.

Subset Construction

NFA $N = (Q, \Sigma, s, \delta, A)$. We create a DFA $M = (Q', \Sigma, \delta', s', A')$ as follows:

- $Q' = \mathcal{P}(Q)$

Subset Construction

NFA $N = (Q, \Sigma, s, \delta, A)$. We create a DFA $M = (Q', \Sigma, \delta', s', A')$ as follows:

- $Q' = \mathcal{P}(Q)$
- $s' = \epsilon\text{reach}(s) = \delta^*(s, \epsilon)$

Subset Construction

NFA $N = (Q, \Sigma, s, \delta, A)$. We create a DFA $M = (Q', \Sigma, \delta', s', A')$ as follows:

- $Q' = \mathcal{P}(Q)$
- $s' = \epsilon\text{reach}(s) = \delta^*(s, \epsilon)$
- $A' = \{X \subseteq Q \mid X \cap A \neq \emptyset\}$

Subset Construction

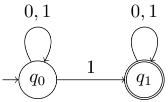
NFA $N = (Q, \Sigma, s, \delta, A)$. We create a DFA $M = (Q', \Sigma, \delta', s', A')$ as follows:

- $Q' = \mathcal{P}(Q)$
- $s' = \epsilon\text{reach}(s) = \delta^*(s, \epsilon)$
- $A' = \{X \subseteq Q \mid X \cap A \neq \emptyset\}$
- $\delta'(X, a) = \cup_{q \in X} \delta^*(q, a)$ for each $X \subseteq Q$, $a \in \Sigma$.

$\cup \{q_1, q_2, \ldots q_k\}$ $\qquad X \in Q$

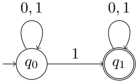
Example

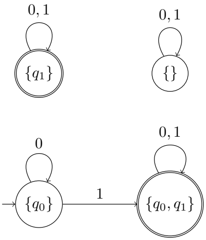
No ϵ-transitions



Example

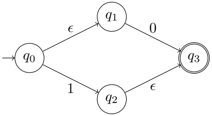
No ϵ-transitions

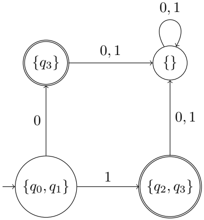




Incremental construction

Only build states reachable from $s' = \epsilon\text{reach}(s)$ the start state of M





$$\delta'(X, a) = \cup_{q \in X} \delta^*(q, a)$$

Incremental algorithm

- Build M beginning with start state $s' == \epsilon\text{reach}(s)$
- For each existing state $X \subseteq Q$ consider each $a \in \Sigma$ and calculate the state $Y = \delta'(X, a) = \cup_{q \in X} \delta^*(q, a)$ and add a transition.
- If Y is a new state add it to reachable states that need to explored.

To compute $\delta^*(q, a)$ - set of all states reached from q on *string* a

- Compute $X = \epsilon\text{reach}(q)$
- Compute $Y = \cup_{p \in X} \delta(p, a)$
- Compute $Z = \epsilon\text{reach}(Y) = \cup_{r \in Y} \epsilon\text{reach}(r)$

Proof of Correctness

Theorem

Let $N = (Q, \Sigma, s, \delta, A)$ *be a NFA and let* $M = (Q', \Sigma, \delta', s', A')$ *be a DFA constructed from* N *via the subset construction. Then* $L(N) = L(M)$.

Proof of Correctness

Theorem

Let $N = (Q, \Sigma, s, \delta, A)$ *be a NFA and let* $M = (Q', \Sigma, \delta', s', A')$ *be a DFA constructed from* N *via the subset construction. Then* $L(N) = L(M)$.

Stronger claim:

Lemma

For every string w, $\delta^*_N(s, w) = \delta^*_M(s', w)$.

Proof by induction on $|w|$.

Base case: $w = \epsilon$.
$\delta^*_N(s, \epsilon) = \epsilon\text{reach}(s)$.
$\delta^*_M(s', \epsilon) = s' = \epsilon\text{reach}(s)$ by definition of s'.

Proof continued

Lemma

For every string w, $\delta_N^*(s, w) = \delta_M^*(s', w)$.

Inductive step: $w = xa$ (Note: suffix definition of strings)

$\delta_N^*(s, xa) = \cup_{p \in \delta_N^*(s,x)} \delta_N^*(p, a)$ by inductive defn of δ_N^*

Proof continued

Lemma

For every string w, $\delta^*_N(s, w) = \delta^*_M(s', w)$.

Inductive step: $w = xa$ (Note: suffix definition of strings)

$\delta^*_N(s, xa) = \cup_{p \in \delta^*_N(s,x)} \delta^*_N(p, a)$ by inductive defn of δ^*_N

$\delta^*_M(s', xa) = \delta_M(\delta^*_M(s, x), a)$ by inductive defn of δ^*_M

Proof continued

Lemma

For every string w, $\delta_N^*(s, w) = \delta_M^*(s', w)$.

Inductive step: $w = xa$ (Note: suffix definition of strings)

$\delta_N^*(s, xa) = \cup_{p \in \delta_N^*(s,x)} \delta_N^*(p, a)$ by inductive defn of δ_N^*

$\delta_M^*(s', xa) = \delta_M(\delta_M^*(s, x), a)$ by inductive defn of δ_M^*

By inductive hypothesis: $Y = \delta_N^*(s, x) = \delta_M^*(s, x)$

Proof continued

Lemma

For every string w, $\delta^*_N(s, w) = \delta^*_M(s', w)$.

Inductive step: $w = xa$ (Note: suffix definition of strings)
$\delta^*_N(s, xa) = \cup_{p \in \delta^*_N(s,x)} \delta^*_N(p, a)$ by inductive defn of δ^*_N
$\delta^*_M(s', xa) = \delta_M(\delta^*_M(s, x), a)$ by inductive defn of δ^*_M

By inductive hypothesis: $Y = \delta^*_N(s, x) = \delta^*_M(s, x)$

Thus $\delta^*_N(s, xa) = \cup_{p \in Y} \delta^*_N(p, a) = \delta_M(Y, a)$ by definition of δ_M.

Proof continued

Lemma

For every string w, $\delta_N^*(s, w) = \delta_M^*(s', w)$.

Inductive step: $w = xa$ (Note: suffix definition of strings)
$\delta_N^*(s, xa) = \cup_{p \in \delta_N^*(s,x)} \delta_N^*(p, a)$ by inductive defn of δ_N^*
$\delta_M^*(s', xa) = \delta_M(\delta_M^*(s, x), a)$ by inductive defn of δ_M^*

By inductive hypothesis: $Y = \delta_N^*(s, x) = \delta_M^*(s, x)$

Thus $\delta_N^*(s, xa) = \cup_{p \in Y} \delta_N^*(p, a) = \delta_M(Y, a)$ by definition of δ_M.

Therefore,
$\delta_N^*(s, xa) = \delta_M(Y, a) = \delta_M(\delta_M^*(s, x), a) = \delta_M^*(s', xa)$
which is what we need.

Part II

Closure Properties of Regular Languages

Regular Languages

Regular languages have three different characterizations

- Inductive definition via base cases and closure under union, concatenation and Kleene star
- Languages accepted by DFAs
- Languages accepted by NFAs

Regular Languages

Regular languages have three different characterizations

- Inductive definition via base cases and closure under union, concatenation and Kleene star
- Languages accepted by DFAs
- Languages accepted by NFAs

Regular language closed under many operations:

- union, concatenation, Kleene star via inductive definition or NFAs
- complement, union, intersection via DFAs
- homomorphism, inverse homomorphism, reverse, ...

Different representations allow for flexibility in proofs

Example: PREFIX

Let L be a language over Σ.

Definition

$\mathrm{PREFIX}(L) = \{w \mid wx \in L, x \in \Sigma^*\}$

Example: PREFIX

Let L be a language over Σ.

Definition

$\text{PREFIX}(L) = \{w \mid wx \in L, x \in \Sigma^*\}$

Theorem

If L is regular then $\text{PREFIX}(L)$ is regular.

Example: PREFIX

Let L be a language over Σ.

Definition

$\text{PREFIX}(L) = \{w \mid wx \in L, x \in \Sigma^*\}$

Theorem

If L is regular then $\text{PREFIX}(L)$ is regular.

Let $M = (Q, \Sigma, \delta, s, A)$ be a DFA that recognizes L

Example: PREFIX

Let L be a language over Σ.

Definition

$\text{PREFIX}(L) = \{w \mid wx \in L, x \in \Sigma^*\}$

Theorem

If L is regular then PREFIX(L) is regular.

Let $M = (Q, \Sigma, \delta, s, A)$ be a DFA that recognizes L
$X = \{q \in Q \mid s \text{ can reach } q \text{ in } M\}$

Example: PREFIX

Let L be a language over Σ.

Definition

$\text{PREFIX}(L) = \{w \mid wx \in L, x \in \Sigma^*\}$

Theorem

If L is regular then $\text{PREFIX}(L)$ is regular.

Let $M = (Q, \Sigma, \delta, s, A)$ be a DFA that recognizes L
$X = \{q \in Q \mid s \text{ can reach } q \text{ in } M\}$
$Y = \{q \in Q \mid q \text{ can reach some state in } A\}$

Example: PREFIX

Let L be a language over Σ.

Definition

$\text{PREFIX}(L) = \{w \mid wx \in L, x \in \Sigma^*\}$

Theorem

If L is regular then $PREFIX(L)$ is regular.

Let $M = (Q, \Sigma, \delta, s, A)$ be a DFA that recognizes L
$X = \{q \in Q \mid s \text{ can reach } q \text{ in } M\}$
$Y = \{q \in Q \mid q \text{ can reach some state in } A\}$
$Z = X \cap Y$
Create new DFA $M' = (Q, \Sigma, \delta, s, Z)$

Example: PREFIX

Let L be a language over Σ.

Definition

$\text{PREFIX}(L) = \{w \mid wx \in L, x \in \Sigma^*\}$

Theorem

If L is regular then $PREFIX(L)$ is regular.

Let $M = (Q, \Sigma, \delta, s, A)$ be a DFA that recognizes L
$X = \{q \in Q \mid s \text{ can reach } q \text{ in } M\}$
$Y = \{q \in Q \mid q \text{ can reach some state in } A\}$
$Z = X \cap Y$
Create new DFA $M' = (Q, \Sigma, \delta, s, Z)$
Claim: $L(M) = \text{PREFIX}(L)$.

Exercise: SUFFIX

Let L be a language over Σ.

Definition

$\text{SUFFIX}(L) = \{w \mid xw \in L, x \in \Sigma^*\}$

Prove the following:

Theorem

If L is regular then $PREFIX(L)$ is regular.