For each of the following languages over the alphabet $\Sigma = \{\mathtt{0}, \mathtt{1}\}$, either prove the language is regular (by giving an equivalent regular expression, DFA, or NFA) or prove that the language is not regular (using a fooling set argument). Exactly half of these languages are regular.

1. $\{\mathtt{0}^n\mathtt{1}\mathtt{0}^n \mid n \geq 0\}$

   Solution: Not regular: Any two strings $x = \mathtt{0}^i$ and $y = \mathtt{0}^j$ are distinguished by the suffix $z = \mathtt{1}\mathtt{0}^i$. Thus, $\mathtt{0}^*$ is a fooling set. ■

2. $\{\mathtt{0}^n\mathtt{1}\mathtt{0}^n w \mid n \geq 0 \text{ and } w \in \Sigma^*\}$

   Solution: Not regular. Any two strings $x = \mathtt{0}^i$ and $y = \mathtt{0}^j$ where $i < j$ are distinguished by the suffix $z = \mathtt{1}\mathtt{0}^i$. (It is crucial that $i < j$ here!) Thus, $\mathtt{0}^*$ is a fooling set. ■

3. $\{w\mathtt{0}^n\mathtt{1}\mathtt{0}^n x \mid w \in \Sigma^* \text{ and } n \geq 0 \text{ and } x \in \Sigma^*\}$

   Solution: Regular. This is the set of all strings containing the symbol $\mathtt{1}$, which is described by the regular expression $\mathtt{0}^*\mathtt{1}(\mathtt{0}+\mathtt{1})^*$. ■

4. Strings in which the number of $\mathtt{0}$s and the number of $\mathtt{1}$s differ by at most 2.

   Solution: Not regular. Any two strings $x = \mathtt{0}^i$ and $y = \mathtt{0}^j$ where $i < j$ are distinguished by the suffix $z = \mathtt{1}^{j+2}$. (It is crucial that $i < j$ here!) Thus, $\mathtt{0}^*$ is a fooling set. ■

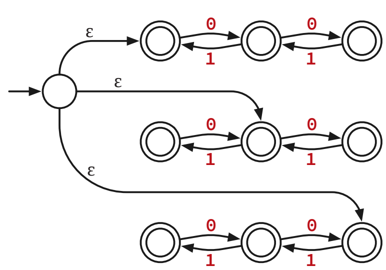
5. Strings such that *in every prefix*, the number of $\mathtt{0}$s and the number of $\mathtt{1}$s differ by at most 2.

   Solution: Regular. Keep track of the difference between the number of $\mathtt{0}$s and the number of $\mathtt{1}$s seen so far. If this difference is ever less than -2 or greater than 2, reject; otherwise, accept. So we get a six-state DFA, where five of the states are accepting. ■

6. Strings such that *in every substring*, the number of $\mathtt{0}$s and the number of $\mathtt{1}$s differ by at most 2.

   Solution: Regular. Keep track of the *current* difference between the number of $\mathtt{0}$s and the number of $\mathtt{1}$s seen so far. Also keep track of the *maximum* and *minimum* value of this difference seen so far. If the max-difference is ever more than min-difference+2, reject. Crudely, there are at most 45 possible values of (curr-dif, max-diff, min-diff), so we get a DFA with at most 46 states.

   Alternatively, we can non-deterministically guess the range of differences ($-2 \leq \mathit{diff} \leq 0$ or $-1 \leq \mathit{diff} \leq 1$ or $0 \leq \mathit{diff} \leq 2$), build a separate DFA for each guess, and combine the three DFAs into a single 10-state NFA.



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