

Prove that each of the following languages is *not* regular.

1. $\{0^{2^n} \mid n \geq 0\}$

Solution (verbose): Let $F = L = \{0^{2^n} \mid n \geq 0\}$.

Let x and y be arbitrary elements of F .

Then $x = 0^{2^i}$ and $y = 0^{2^j}$ for some non-negative integers x and y .

Let $z = 0^{2^i}$.

Then $xz = 0^{2^i} 0^{2^i} = 0^{2^{i+1}} \in L$.

And $yz = 0^{2^j} 0^{2^i} = 0^{2^i+2^j} \notin L$, because $i \neq j$

Thus, F is a fooling set for L .

Because F is infinite, L cannot be regular. ■

Solution (concise): For any non-negative integers $i \neq j$, the strings 0^{2^i} and 0^{2^j} are distinguished by the suffix 0^{2^i} , because $0^{2^i} 0^{2^i} = 0^{2^{i+1}} \in L$ but $0^{2^j} 0^{2^i} = 0^{2^i+2^j} \notin L$. Thus L itself is an infinite fooling set for L . ■

2. $\{0^{2^n} 1^n \mid n \geq 0\}$

Solution (verbose): Let F be the language 0^* .

Let x and y be arbitrary strings in F .

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Let $z = 0^i 1^i$.

Then $xz = 0^{2i} 1^i \in L$.

And $yz = 0^{i+j} 1^i \notin L$, because $i + j \neq 2i$.

Thus, F is a fooling set for L .

Because F is infinite, L cannot be regular. ■

Solution (concise): For all non-negative integers $i \neq j$, the strings 0^i and 0^j are distinguished by the suffix $0^i 1^i$, because $0^{2i} 1^i \in L$ but $0^{i+j} 1^i \notin L$. Thus, the language 0^* is an infinite fooling set for L . ■

Solution (concise, different fooling set): For all non-negative integers $i \neq j$, the strings 0^{2i} and 0^{2j} are distinguished by the suffix 1^i , because $0^{2i} 1^i \in L$ but $0^{2j} 1^i \notin L$. Thus, the language $(00)^*$ is an infinite fooling set for L . ■

3. $\{0^m 1^n \mid m \neq 2n\}$

Solution (verbose): Let F be the language 0^* .

Let x and y be arbitrary strings in F .

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Let $z = 0^i 1^i$.

Then $xz = 0^{2i} 1^i \notin L$.

And $yz = 0^{i+j} 1^i \in L$, because $i + j \neq 2i$.

Thus, F is a fooling set for L .

Because F is infinite, L cannot be regular. ■

Solution (concise, different fooling set): For all non-negative integers $i \neq j$, the strings 0^{2i} and 0^{2j} are distinguished by the suffix 1^i , because $0^{2i} 1^i \notin L$ but $0^{2j} 1^i \in L$. Thus, the language $(00)^*$ is an infinite fooling set for L . ■

4. Strings over $\{0, 1\}$ where the number of 0s is exactly twice the number of 1s.

Solution (verbose): Let F be the language 0^* .

Let x and y be arbitrary strings in F .

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Let $z = 0^i 1^i$.

Then $xz = 0^{2i} 1^i \in L$.

And $yz = 0^{i+j} 1^i \notin L$, because $i + j \neq 2i$.

Thus, F is a fooling set for L .

Because F is infinite, L cannot be regular. ■

Solution (concise, different fooling set): For all non-negative integers $i \neq j$, the strings 0^{2i} and 0^{2j} are distinguished by the suffix 1^i , because $0^{2i} 1^i \in L$ but $0^{2j} 1^i \notin L$. Thus, the language $(00)^*$ is an infinite fooling set for L . ■

Solution (closure properties): If L were regular, then the language

$$((0 + 1)^* \setminus L) \cap 0^* 1^* = \{0^m 1^n \mid m \neq 2n\}$$

would also be regular, because regular languages are closed under complement and intersection. But we just proved that $\{0^m 1^n \mid m \neq 2n\}$ is not regular in problem 3. [Yes, this proof would be worth full credit, either in homework or on an exam.] ■

5. Strings of properly nested parentheses $()$, brackets $[\]$, and braces $\{\}$. For example, the string $([\])\{\}$ is in this language, but the string $([\])$ is not, because the left and right delimiters don't match.

Solution (verbose): Let F be the language $(^*$.

Let x and y be arbitrary strings in F .

Then $x = ($ ^{i} and $y = ($ ^{j} for some non-negative integers $i \neq j$.

Let $z =)$ ^{i} .

Then $xz = ($ ^{i} $)$ ^{i} $\in L$.

And $yz = ($ ^{j} $)$ ^{i} $\notin L$, because $i \neq j$.

Thus, F is a fooling set for L .

Because F is infinite, L cannot be regular. ■

Solution (concise): For any non-negative integers $i \neq j$, the strings $($ ^{i} and $($ ^{j} are distinguished by the suffix $)$ ^{i} , because $($ ^{i} $)$ ^{i} $\in L$ but $($ ^{i} $)$ ^{j} $\notin L$. Thus, the language $(^*$ is an infinite fooling set. ■

6. Strings of the form $w_1\#w_2\#\dots\#w_n$ for some $n \geq 2$, where each substring w_i is a string in $\{0, 1\}^*$, and some pair of substrings w_i and w_j are equal.

Solution (verbose): Let F be the language 0^* .

Let x and y be arbitrary strings in F .

Then $x = 0$ ^{i} and $y = 0$ ^{j} for some non-negative integers $i \neq j$.

Let $z = \#0$ ^{i} .

Then $xz = 0$ ^{i} $\#0$ ^{i} $\in L$.

And $yz = 0$ ^{j} $\#0$ ^{i} $\notin L$, because $i \neq j$.

Thus, F is a fooling set for L .

Because F is infinite, L cannot be regular. ■

Solution (concise): For any non-negative integers $i \neq j$, the strings 0 ^{i} and 0 ^{j} are distinguished by the suffix $\#0$ ^{i} , because 0 ^{i} $\#0$ ^{i} $\in L$ but 0 ^{j} $\#0$ ^{i} $\notin L$. Thus, the language 0^* is an infinite fooling set. ■

Work on these later:

7. $\{\emptyset^{n^2} \mid n \geq 0\}$

Solution: Let x and y be distinct arbitrary strings in L .

Without loss of generality, $x = \emptyset^{i^2}$ and $y = \emptyset^{j^2}$ for some $j > i \geq 0$.

Let $z = \emptyset^{2i+1}$.

Then $xz = \emptyset^{i^2+2i+1} = \emptyset^{(i+1)^2} \in L$

On the other hand, $yz = \emptyset^{j^2+2i+1} \notin L$, because $j^2 < j^2 + 2i + 1 < j^2 + 2j + 1 = (j + 1)^2$.

Thus, z distinguishes x and y .

We conclude that L is an infinite fooling set for L , so L cannot be regular. ■

Solution: Let x and y be distinct arbitrary strings in \emptyset^* .

Without loss of generality, $x = \emptyset^i$ and $y = \emptyset^j$ for some $i > j \geq 0$.

Let $z = \emptyset^{i^2+i+1}$.

Then $xz = \emptyset^{i^2+2i+1} = \emptyset^{(i+1)^2} \in L$.

On the other hand, $yz = \emptyset^{i^2+i+j+1} \notin L$, because $i^2 < i^2 + i + j + 1 < (i + 1)^2$.

Thus, z distinguishes x and y .

We conclude that \emptyset^* is an infinite fooling set for L , so L cannot be regular. ■

Solution: Let x and y be distinct arbitrary strings in $\emptyset\emptyset\emptyset\emptyset^*$.

Without loss of generality, $x = \emptyset^i$ and $y = \emptyset^j$ for some $i > j \geq 3$.

Let $z = \emptyset^{i^2-i}$.

Then $xz = \emptyset^{i^2} \in L$.

On the other hand, $yz = \emptyset^{i^2-i+j} \notin L$, because

$$(i-1)^2 = i^2 - 2i + 1 < i^2 - i < i^2 - i + j < i^2.$$

(The first inequality requires $i \geq 2$, and the second $j \geq 1$.)

Thus, z distinguishes x and y .

We conclude that $\emptyset\emptyset\emptyset\emptyset^*$ is an infinite fooling set for L , so L cannot be regular. ■

8. $\{w \in (\mathbf{0} + \mathbf{1})^* \mid w \text{ is the binary representation of a perfect square}\}$

Solution: We design our fooling set around numbers of the form $(2^k + 1)^2 = 2^{2k} + 2^{k+1} + 1 = \mathbf{10}^{k-2}\mathbf{10}^k\mathbf{1} \in L$, for any integer $k \geq 2$. The argument is somewhat simpler if we further restrict k to be even.

Let $F = \mathbf{1}(\mathbf{00})^*\mathbf{1}$, and let x and y be arbitrary strings in F .

Then $x = \mathbf{10}^{2i-2}\mathbf{1}$ and $y = \mathbf{10}^{2j-2}\mathbf{1}$, for some positive integers $i \neq j$.

Without loss of generality, assume $i < j$. (Otherwise, swap x and y .)

Let $z = \mathbf{0}^{2i}\mathbf{1}$.

Then $xz = \mathbf{10}^{2i-2}\mathbf{10}^{2i}\mathbf{1}$ is the binary representation of $2^{4i} + 2^{2i+1} + 1 = (2^{2i} + 1)^2$, and therefore $xz \in L$.

On the other hand, $yz = \mathbf{10}^{2j-2}\mathbf{10}^{2i}\mathbf{1}$ is the binary representation of $2^{2i+2j} + 2^{2i+1} + 1$. Simple algebra gives us the inequalities

$$\begin{aligned} (2^{i+j})^2 &= 2^{2i+2j} \\ &< 2^{2i+2j} + 2^{2i+1} + 1 \\ &< 2^{2(i+j)} + 2^{i+j+1} + 1 \\ &= (2^{i+j} + 1)^2. \end{aligned}$$

So $2^{2i+2j} + 2^{2i+1} + 1$ lies between two consecutive perfect squares, and thus is not a perfect square, which implies that $yz \notin L$.

We conclude that F is a fooling set for L . Because F is infinite, L cannot be regular. ■