Breadth First Search, Dijkstra's Algorithm for Shortest Paths

Lecture 16

October 20, 2015

Part I

Breadth First Search

Breadth First Search (BFS)

Overview

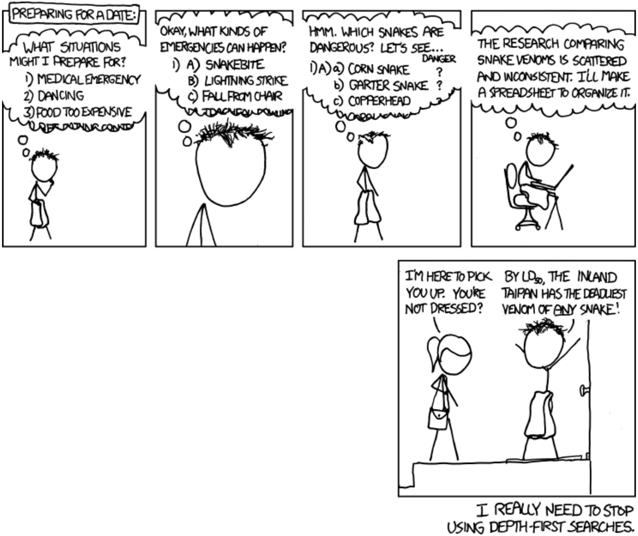
(A) **BFS** is obtained from **BasicSearch** by processing edges using a data structure called a **queue**.

(B) It processes the vertices in the graph in the order of their shortest distance from the vertex **s** (the start vertex).

As such...

1. **DFS** good for exploring graph structure
2. **BFS** good for exploring *distances*

xkcd take on DFS



Queue Data Structure

Queues

A **queue** is a list of elements which supports the operations:

1. **enqueue**: Adds an element to the end of the list
2. **dequeue**: Removes an element from the front of the list

Elements are extracted in **first-in first-out (FIFO)** order, i.e., elements are picked in the order in which they were inserted.

BFS Algorithm

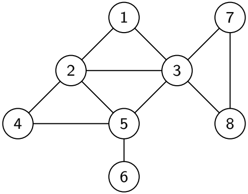
Given (undirected or directed) graph $\mathbf{G = (V, E)}$ and node $\mathbf{s \in V}$

```
BFS(s)
    Mark all vertices as unvisited
    Initialize search tree T to be empty
    Mark vertex s as visited
    set Q to be the empty queue
    enq(s)
    while Q is nonempty do
        u = deq(Q)
        for each vertex v ∈ Adj(u)
            if v is not visited then
                add edge (u, v) to T
                Mark v as visited and enq(v)
```

Proposition

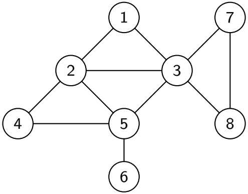
BFS(s) *runs in* $\mathbf{O(n + m)}$ *time.*

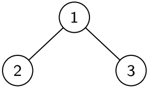
BFS: An Example in Undirected Graphs



1. [1]

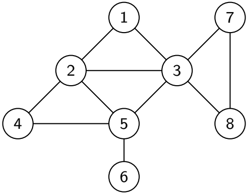
BFS: An Example in Undirected Graphs

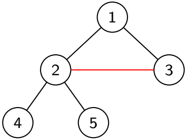




1. [1]
2. [2,3]

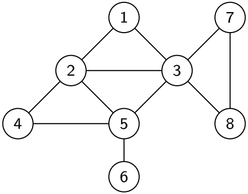
BFS: An Example in Undirected Graphs

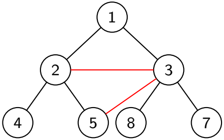




1. [1]
2. [2,3]
3. [3,4,5]

BFS: An Example in Undirected Graphs





1. [1]
2. [2,3]
3. [3,4,5]
4. [4,5,7,8]

BFS: An Example in Undirected Graphs

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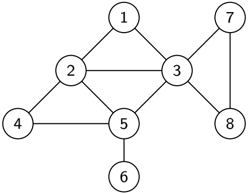
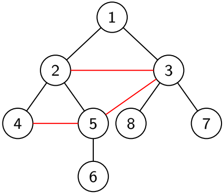
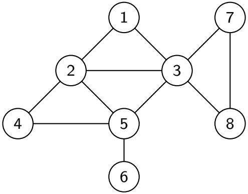
BFS: An Example in Undirected Graphs

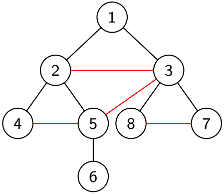




1. [1]
2. [2,3]
3. [3,4,5]
4. [4,5,7,8]
5. [5,7,8]
6. [7,8,6]

BFS: An Example in Undirected Graphs





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6. [7,8,6]
7. [8,6]

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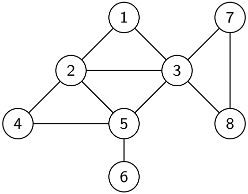

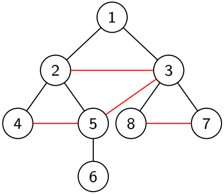

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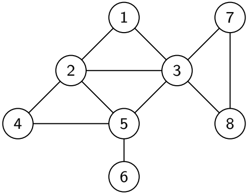
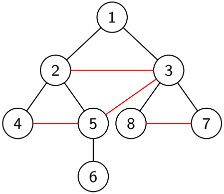
BFS: An Example in Undirected Graphs





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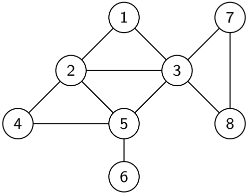
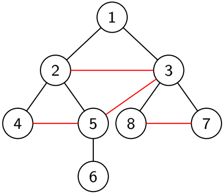
3. [3,4,5]
4. [4,5,7,8]
5. [5,7,8]
6. [7,8,6]
7. [8,6]
8. [6]
9. []

BFS: An Example in Undirected Graphs

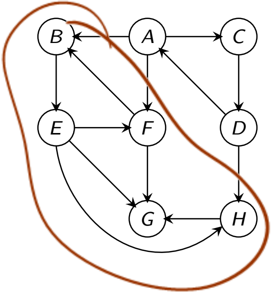


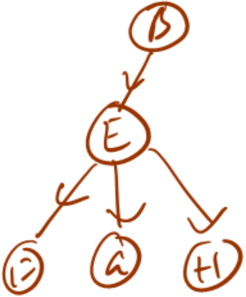


1.	[1]	4.	[4,5,7,8]	7.	[8,6]
2.	[2,3]	5.	[5,7,8]	8.	[6]
3.	[3,4,5]	6.	[7,8,6]	9.	[]

BFS tree is the set of black edges.

BFS: An Example in Directed Graphs





BFS with Distance

```
BFS(s)
    Mark all vertices as unvisited; for each v set dist(v) = ∞
    Initialize search tree T to be empty
    Mark vertex s as visited and set dist(s) = 0
    set Q to be the empty queue
    enq(s)
    while Q is nonempty do
        u = deq(Q)
        for each vertex v ∈ Adj(u) do
            if v is not visited do
                add edge (u, v) to T
                Mark v as visited, enq(v)
                and set dist(v) = dist(u) + 1
```

Properties of BFS: Undirected Graphs

Theorem

The following properties hold upon termination of **BFS**$(\mathbf{s})$

(A) *The search tree contains exactly the set of vertices in the connected component of* $\mathbf{s}$.

(B) *If* $\mathrm{dist}(\mathbf{u}) < \mathrm{dist}(\mathbf{v})$ *then* $\mathbf{u}$ *is visited before* $\mathbf{v}$.

(C) *For every vertex* $\mathbf{u}$, $\mathrm{dist}(\mathbf{u})$ *is the length of a shortest path (in terms of number of edges) from* $\mathbf{s}$ *to* $\mathbf{u}$.

(D) *If* $\mathbf{u}, \mathbf{v}$ *are in connected component of* $\mathbf{s}$ *and* $\mathbf{e} = \{\mathbf{u}, \mathbf{v}\}$ *is an edge of* $\mathbf{G}$, *then* $|\mathrm{dist}(\mathbf{u}) - \mathrm{dist}(\mathbf{v})| \leq 1$.

Properties of BFS: Directed Graphs

Theorem

The following properties hold upon termination of **BFS**$(\mathbf{s})$*:*

(A) *The search tree contains exactly the set of vertices reachable from* $\mathbf{s}$

(B) *If* $\mathrm{dist}(\mathbf{u}) < \mathrm{dist}(\mathbf{v})$ *then* $\mathbf{u}$ *is visited before* $\mathbf{v}$

(C) *For every vertex* $\mathbf{u}$, $\mathrm{dist}(\mathbf{u})$ *is indeed the length of shortest path from* $\mathbf{s}$ *to* $\mathbf{u}$

(D) *If* $\mathbf{u}$ *is reachable from* $\mathbf{s}$ *and* $\mathbf{e} = (\mathbf{u}, \mathbf{v})$ *is an edge of* $\mathbf{G}$*, then* $\mathrm{dist}(\mathbf{v}) - \mathrm{dist}(\mathbf{u}) \leq \mathbf{1}$.
Not necessarily the case that $\mathrm{dist}(\mathbf{u}) - \mathrm{dist}(\mathbf{v}) \leq \mathbf{1}$.

BFS with Layers

```
BFSLayers(s):
    Mark all vertices as unvisited and initialize T to be empty
    Mark s as visited and set L_0 = {s}
    i = 0
    while L_i is not empty do
            initialize L_{i+1} to be an empty list
            for each u in L_i do
                for each edge (u, v) ∈ Adj(u) do
                if v is not visited
                        mark v as visited
                        add (u, v) to tree T
                        add v to L_{i+1}
            i = i + 1
```

BFS with Layers

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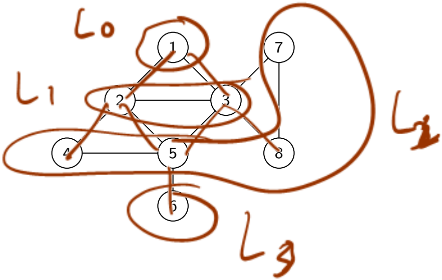
```

Running time: $O(n + m)$

Example



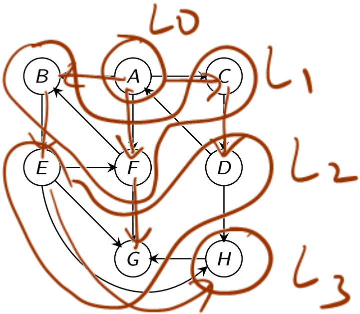
BFS with Layers: Properties

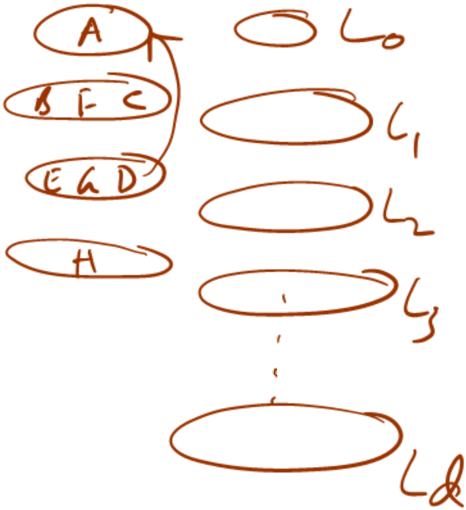
Proposition

The following properties hold on termination of **BFSLayers**$(\mathbf{s})$.

1. **BFSLayers**$(\mathbf{s})$ *outputs a* **BFS** *tree*
2. $\mathbf{L_i}$ *is the set of vertices at distance exactly* $\mathbf{i}$ *from* $\mathbf{s}$
3. *If* $\mathbf{G}$ *is undirected, each edge* $\mathbf{e} = \{\mathbf{u}, \mathbf{v}\}$ *is one of three types:*
    1. **tree** *edge between two* consecutive *layers*
    2. *non-tree* **forward**/**backward** *edge between two consecutive layers*
    3. *non-tree* **cross-edge** *with both* $\mathbf{u}, \mathbf{v}$ *in same layer*
    4. $\implies$ *Every edge in the graph is either between two vertices that are either (i) in the same layer, or (ii) in two consecutive layers.*

Example





BFS with Layers: Properties

For directed graphs

Proposition

The following properties hold on termination of **BFSLayers**(s)*, if* G *is directed.*

For each edge $e = (u, v)$ *is one of four types:*

1. *a* **tree** *edge between consecutive layers,* $u \in L_i, v \in L_{i+1}$ *for some* $i \geq 0$
2. *a non-tree* **forward** *edge between consecutive layers*
3. *a non-tree* **backward** *edge*
4. *a* **cross-edge** *with both* u, v *in same layer*

Part II

Shortest Paths and Dijkstra's Algorithm

Shortest Path Problems

Shortest Path Problems

Input A (undirected or directed) graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ with edge lengths (or costs). For edge $\mathbf{e} = (\mathbf{u}, \mathbf{v})$, $\ell(\mathbf{e}) = \ell(\mathbf{u}, \mathbf{v})$ is its length.

1. Given nodes $\mathbf{s}, \mathbf{t}$ find shortest path from $\mathbf{s}$ to $\mathbf{t}$.
2. Given node $\mathbf{s}$ find shortest path from $\mathbf{s}$ to all other nodes.
3. Find shortest paths for all pairs of nodes.

Shortest Path Problems

Input A (undirected or directed) graph $G = (V, E)$ with edge lengths (or costs). For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

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Many applications!

Single-Source Shortest Paths:

Non-Negative Edge Lengths

Single-Source Shortest Path Problems

1. Input: A (undirected or directed) graph $G = (V, E)$ with non-negative edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.
2. Given nodes s, t find shortest path from s to t.
3. Given node s find shortest path from s to all other nodes.

Single-Source Shortest Paths:

Non-Negative Edge Lengths

Single-Source Shortest Path Problems

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1. Restrict attention to directed graphs
2. Undirected graph problem can be reduced to directed graph problem - how?

Single-Source Shortest Path Problems

1. Input: A (undirected or directed) graph $\mathbf{G = (V, E)}$ with non-negative edge lengths. For edge $\mathbf{e = (u, v)}$, $\ell(\mathbf{e}) = \ell(\mathbf{u, v})$ is its length.
2. Given nodes $\mathbf{s, t}$ find shortest path from $\mathbf{s}$ to $\mathbf{t}$.
3. Given node $\mathbf{s}$ find shortest path from $\mathbf{s}$ to all other nodes.

1. Restrict attention to directed graphs
2. Undirected graph problem can be reduced to directed graph problem - how?
   1. Given undirected graph $\mathbf{G}$, create a new directed graph $\mathbf{G'}$ by replacing each edge $\mathbf{\{u, v\}}$ in $\mathbf{G}$ by $\mathbf{(u, v)}$ and $\mathbf{(v, u)}$ in $\mathbf{G'}$.
   2. set $\ell(\mathbf{u, v}) = \ell(\mathbf{v, u}) = \ell(\mathbf{\{u, v\}})$
   3. Exercise: show reduction works. Relies on non-negativity!

Single-Source Shortest Paths via BFS

Special case: All edge lengths are $\mathbf{1}$.

Single-Source Shortest Paths via BFS

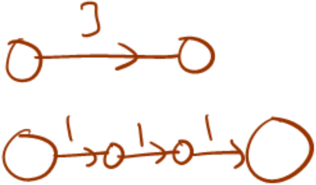
Special case: All edge lengths are 1.

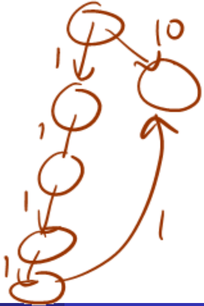
1. Run **BFS**(s) to get shortest path distances from s to all other nodes.
2. $O(m + n)$ time algorithm.

Special case: All edge lengths are $\mathbf{1}$.

1. Run **BFS**($\mathbf{s}$) to get shortest path distances from s to all other nodes.
2. $\mathbf{O(m+n)}$ time algorithm.

Special case: Suppose $\ell(\mathbf{e})$ is an integer for all $\mathbf{e}$? 1, 2, 10
Can we use **BFS**?





Single-Source Shortest Paths via BFS

Special case: All edge lengths are $\mathbf{1}$.

1. Run **BFS**($\mathbf{s}$) to get shortest path distances from s to all other nodes.
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Special case: Suppose $\boldsymbol{\ell(e)}$ is an integer for all $\mathbf{e}$?
Can we use **BFS**? Reduce to unit edge-length problem by placing $\boldsymbol{\ell(e)-1}$ dummy nodes on $\mathbf{e}$

Single-Source Shortest Paths via BFS

Special case: All edge lengths are $\mathbf{1}$.

1. Run **BFS**($\mathbf{s}$) to get shortest path distances from s to all other nodes.
2. $\mathbf{O(m + n)}$ time algorithm.

Special case: Suppose $\boldsymbol{\ell}\mathbf{(e)}$ is an integer for all $\mathbf{e}$?
Can we use **BFS**? Reduce to unit edge-length problem by placing $\boldsymbol{\ell}\mathbf{(e) - 1}$ dummy nodes on $\mathbf{e}$

Let $\mathbf{L = \max_e \ell(e)}$. New graph has $\mathbf{O(mL)}$ edges and $\mathbf{O(mL + n)}$ nodes. **BFS** takes $\mathbf{O(mL + n)}$ time. Not efficient if $\mathbf{L}$ is large.

Towards an algorithm

Why does **BFS** work?

Towards an algorithm

Why does **BFS** work?
BFS(s) explores nodes in increasing distance from **s**

Towards an algorithm

Why does **BFS** work?

BFS(s) explores nodes in increasing distance from $\mathbf{s}$

Lemma

Let $\mathbf{G}$ *be a directed graph with non-negative edge lengths. Let* $\mathrm{dist}(\mathbf{s}, \mathbf{v})$ *denote the shortest path length from* $\mathbf{s}$ *to* $\mathbf{v}$*. If* $\mathbf{s} = \mathbf{v}_0 \to \mathbf{v}_1 \to \mathbf{v}_2 \to \ldots \to \mathbf{v}_k$ *is a shortest path from* $\mathbf{s}$ *to* $\mathbf{v}_k$ *then for* $1 \leq \mathbf{i} < \mathbf{k}$*:*

1. $\mathbf{s} = \mathbf{v}_0 \to \mathbf{v}_1 \to \mathbf{v}_2 \to \ldots \to \mathbf{v}_i$ *is a shortest path from* $\mathbf{s}$ *to* $\mathbf{v}_i$
2. $\mathrm{dist}(\mathbf{s}, \mathbf{v}_i) \leq \mathrm{dist}(\mathbf{s}, \mathbf{v}_k)$. *Relies on non-neg edge lengths.*

Towards an algorithm

Lemma

Let $\mathbf{G}$ *be a directed graph with non-negative edge lengths. Let* $\mathrm{dist}(\mathbf{s}, \mathbf{v})$ *denote the shortest path length from* $\mathbf{s}$ *to* $\mathbf{v}$*. If* $\mathbf{s} = \mathbf{v_0} \rightarrow \mathbf{v_1} \rightarrow \mathbf{v_2} \rightarrow \ldots \rightarrow \mathbf{v_k}$ *is a shortest path from* $\mathbf{s}$ *to* $\mathbf{v_k}$ *then for* $1 \leq \mathbf{i} < \mathbf{k}$*:*

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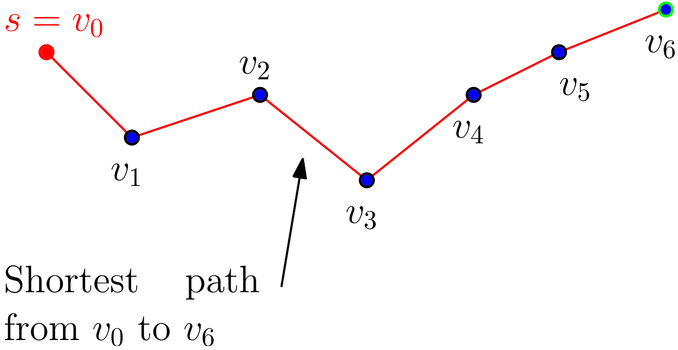
2. $\mathrm{dist}(\mathbf{s}, \mathbf{v_i}) \leq \mathrm{dist}(\mathbf{s}, \mathbf{v_k})$. *Relies on non-neg edge lengths.*

Proof.

Suppose not. Then for some $\mathbf{i} < \mathbf{k}$ there is a path $\mathbf{P'}$ from $\mathbf{s}$ to $\mathbf{v_i}$ of length strictly less than that of $\mathbf{s} = \mathbf{v_0} \rightarrow \mathbf{v_1} \rightarrow \ldots \rightarrow \mathbf{v_i}$. Then $\mathbf{P'}$ concatenated with $\mathbf{v_i} \rightarrow \mathbf{v_{i+1}} \ldots \rightarrow \mathbf{v_k}$ contains a strictly shorter path to $\mathbf{v_k}$ than $\mathbf{s} = \mathbf{v_0} \rightarrow \mathbf{v_1} \ldots \rightarrow \mathbf{v_k}$. For the second part, observe that edge lengths are non-negative. $\square$

A proof by picture



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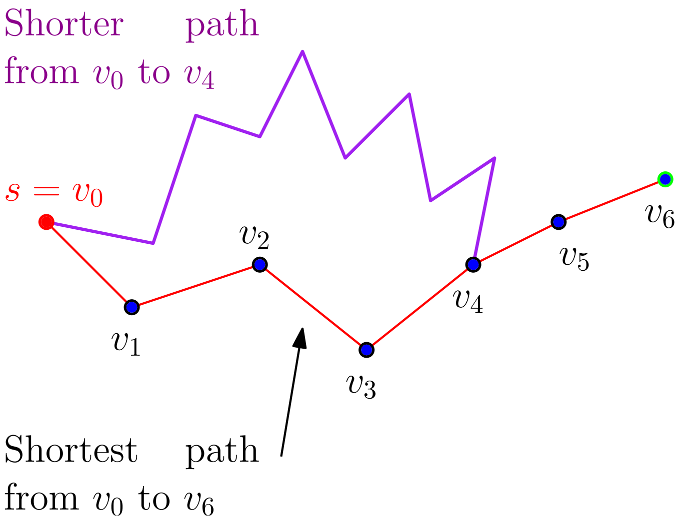

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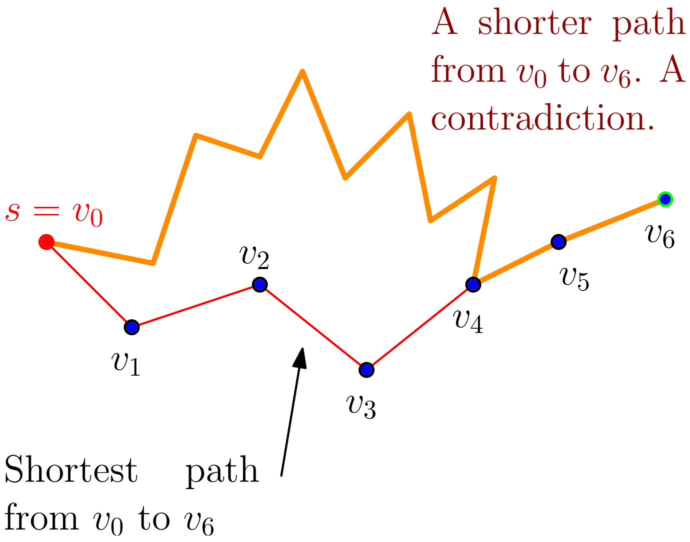

A Basic Strategy

Explore vertices in increasing order of distance from s:
(For simplicity assume that nodes are at different distances from s and that no edge has zero length)

```
Initialize for each node v, dist(s, v) = ∞
Initialize X = {s},
for i = 2 to |V| do
    (* Invariant: X contains the i − 1 closest nodes to s *)
    Among nodes in V − X, find the node v that is the
            i'th closest to s
    Update dist(s, v)
    X = X ∪ {v}
```

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```

How can we implement the step in the for loop?

Finding the ith closest node

1. $\mathbf{X}$ contains the $\mathbf{i-1}$ closest nodes to $\mathbf{s}$
2. Want to find the $\mathbf{i}$th closest node from $\mathbf{V-X}$.

What do we know about the $\mathbf{i}$th closest node?

Finding the ith closest node

1. $\mathbf{X}$ contains the $\mathbf{i-1}$ closest nodes to $\mathbf{s}$
2. Want to find the $\mathbf{i}$th closest node from $\mathbf{V-X}$.

What do we know about the $\mathbf{i}$th closest node?

Claim

Let $\mathbf{P}$ be a shortest path from $\mathbf{s}$ to $\mathbf{v}$ where $\mathbf{v}$ is the $\mathbf{i}$th closest node. Then, all intermediate nodes in $\mathbf{P}$ belong to $\mathbf{X}$.

Finding the ith closest node

1. $\mathbf{X}$ contains the $\mathbf{i}-\mathbf{1}$ closest nodes to $\mathbf{s}$
2. Want to find the $\mathbf{i}$th closest node from $\mathbf{V}-\mathbf{X}$.

What do we know about the $\mathbf{i}$th closest node?

Claim

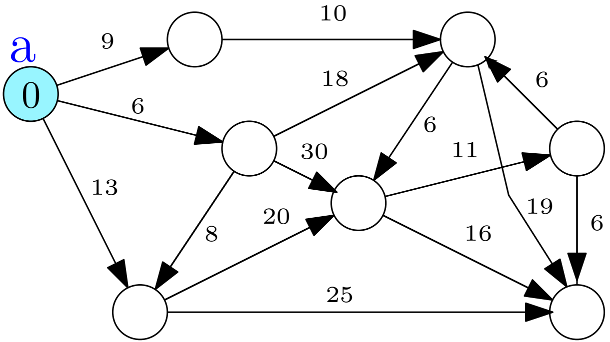
Let $\mathbf{P}$ be a shortest path from $\mathbf{s}$ to $\mathbf{v}$ where $\mathbf{v}$ is the $\mathbf{i}$th closest node. Then, all intermediate nodes in $\mathbf{P}$ belong to $\mathbf{X}$.

Proof.

If $\mathbf{P}$ had an intermediate node $\mathbf{u}$ not in $\mathbf{X}$ then $\mathbf{u}$ will be closer to $\mathbf{s}$ than $\mathbf{v}$. Implies $\mathbf{v}$ is not the $\mathbf{i}$'th closest node to $\mathbf{s}$ - recall that $\mathbf{X}$ already has the $\mathbf{i}-\mathbf{1}$ closest nodes. □

Finding the ith closest node repeatedly

An example



Finding the ith closest node repeatedly

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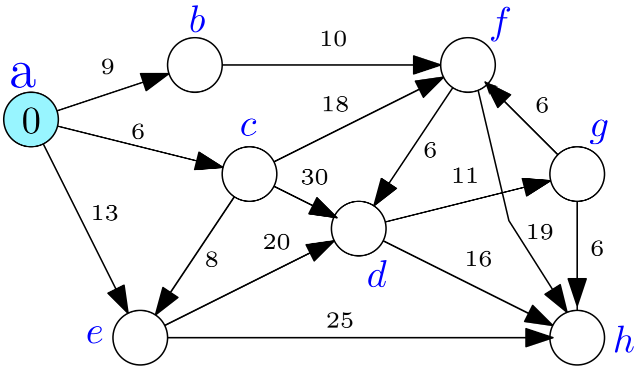

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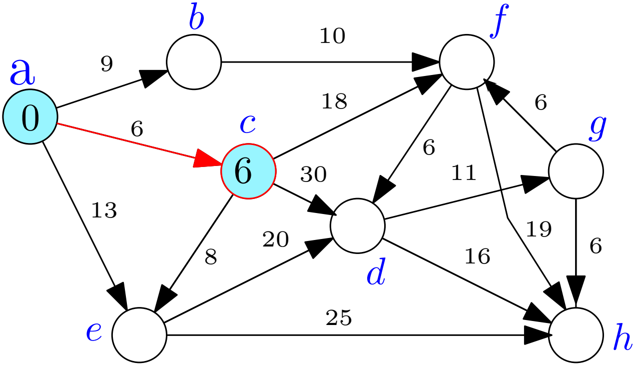

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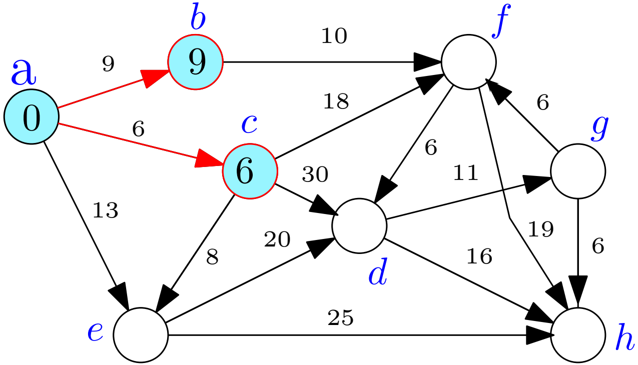

Finding the ith closest node repeatedly

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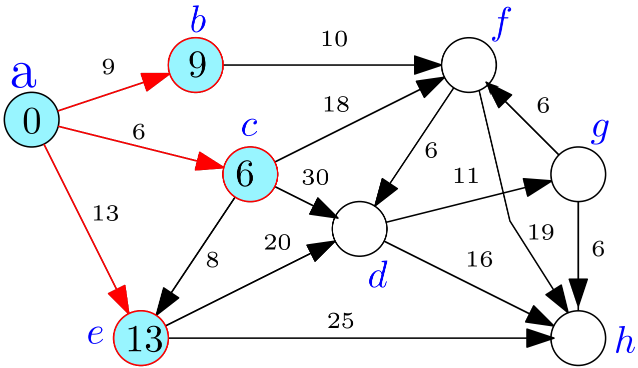
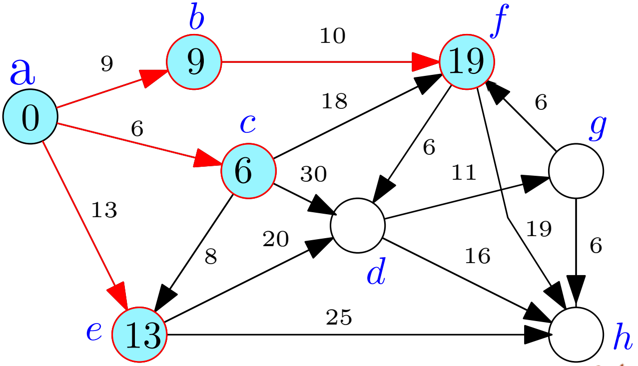


Finding the ith closest node repeatedly

An example

a c b e f d g h

s

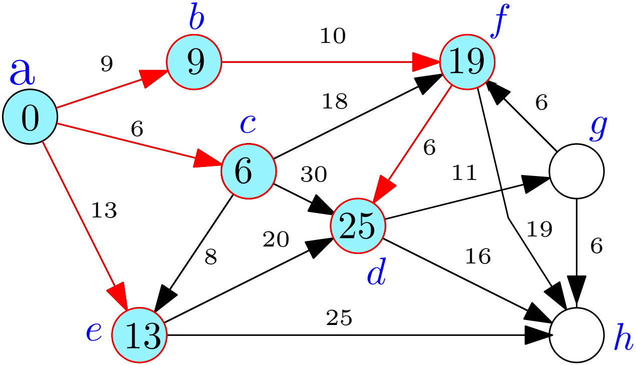


$d'(s,d) = 25$ $d'(s,g) = \infty$

$d'(s,h) = 38$

Finding the ith closest node repeatedly

An example



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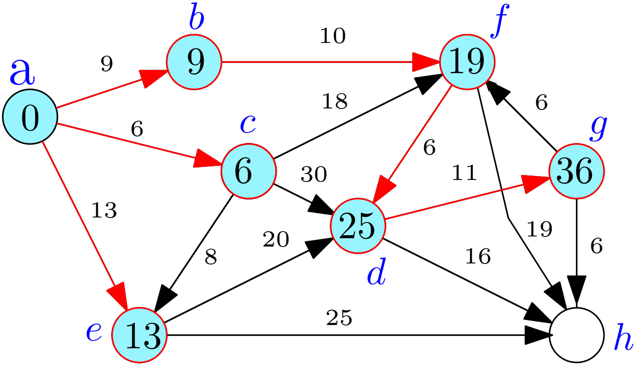

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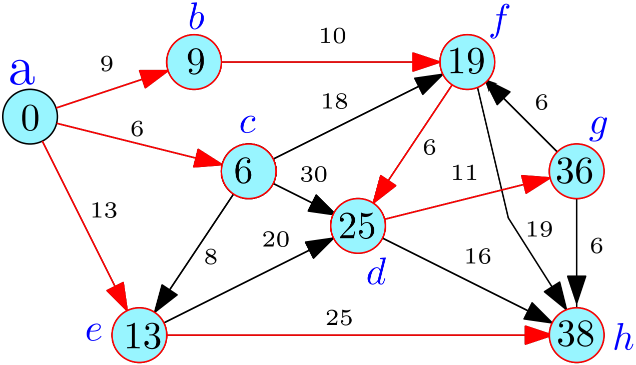

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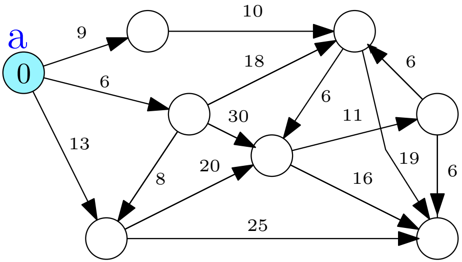

Corollary

The **i***th closest node is adjacent to* **X**.

Finding the ith closest node

1. $\mathbf{X}$ contains the $\mathbf{i-1}$ closest nodes to $\mathbf{s}$
2. Want to find the $\mathbf{i}$th closest node from $\mathbf{V-X}$.

1. For each $\mathbf{u \in V - X}$ let $\mathbf{P(s, u, X)}$ be a shortest path from $\mathbf{s}$ to $\mathbf{u}$ using only nodes in $\mathbf{X}$ as intermediate vertices.
2. Let $\mathbf{d'(s, u)}$ be the length of $\mathbf{P(s, u, X)}$

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Observations: for each $\mathbf{u \in V - S}$,

1. $\mathrm{dist}(\mathbf{s}, \mathbf{u}) \leq \mathbf{d'(s, u)}$ since we are constraining the paths
2. $\mathbf{d'(s, u) = \min_{t \in X}(\mathrm{dist}(s, t) + \ell(t, u))}$ - Why?

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Lemma

If $\mathbf{v}$ is the ith closest node to $\mathbf{s}$, then $\mathbf{d'(s,v)} = \mathrm{dist}(\mathbf{s},\mathbf{v})$.

Finding the ith closest node

Lemma

Given:

1. $\mathbf{X}$: *Set of* $\mathbf{i-1}$ *closest nodes to* $\mathbf{s}$.
2. $\mathbf{d'(s,u) = \min_{t \in X}(\mathrm{dist}(s,t) + \ell(t,u))}$

If $\mathbf{v}$ *is an* $\mathbf{i}$*th closest node to* $\mathbf{s}$*, then* $\mathbf{d'(s,v) = \mathrm{dist}(s,v)}$.

Proof.

Let $\mathbf{v}$ be the $\mathbf{i}$th closest node to $\mathbf{s}$. Then there is a shortest path $\mathbf{P}$ from $\mathbf{s}$ to $\mathbf{v}$ that contains only nodes in $\mathbf{X}$ as intermediate nodes (see previous claim). Therefore $\mathbf{d'(s,v) = \mathrm{dist}(s,v)}$. □

Finding the ith closest node

Lemma

If $\mathbf{v}$ *is an* $\mathbf{i}$*th closest node to* $\mathbf{s}$*, then* $\mathbf{d'(s,v)} = \mathrm{dist}(\mathbf{s},\mathbf{v})$.

Corollary

The $\mathbf{i}$*th closest node to* $\mathbf{s}$ *is the node* $\mathbf{v} \in \mathbf{V} - \mathbf{S}$ *such that* $\mathbf{d'(s,v)} = \mathbf{min}_{\mathbf{u} \in \mathbf{V}-\mathbf{X}} \mathbf{d'(s,u)}$.

Proof.

For every node $\mathbf{u} \in \mathbf{V} - \mathbf{X}$, $\mathrm{dist}(\mathbf{s},\mathbf{u}) \leq \mathbf{d'(s,u)}$ and for the $\mathbf{i}$th closest node $\mathbf{v}$, $\mathrm{dist}(\mathbf{s},\mathbf{v}) = \mathbf{d'(s,v)}$. Moreover, $\mathrm{dist}(\mathbf{s},\mathbf{u}) \geq \mathrm{dist}(\mathbf{s},\mathbf{v})$ for each $\mathbf{u} \in \mathbf{V} - \mathbf{S}$. □

Algorithm

```
Initialize for each node v: dist(s, v) = ∞
Initialize X = ∅, d'(s, s) = 0
for i = 1 to |V| do
    (* Invariant: X contains the i − 1 closest nodes to s *)
    (* Invariant: d'(s, u) is shortest path distance from u to s
     using only X as intermediate nodes*)
    Let v be such that d'(s, v) = min_{u∈V−X} d'(s, u)
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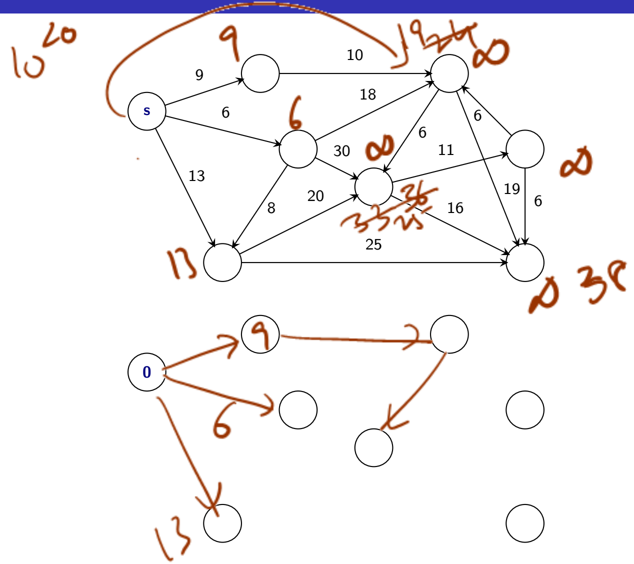
Correctness: By induction on i using previous lemmas.

Running time: $O(n \cdot (n + m))$ time.

1. n outer iterations. In each iteration, $d'(s, u)$ for each u by scanning all edges out of nodes in X; $O(m + n)$ time/iteration.

Example



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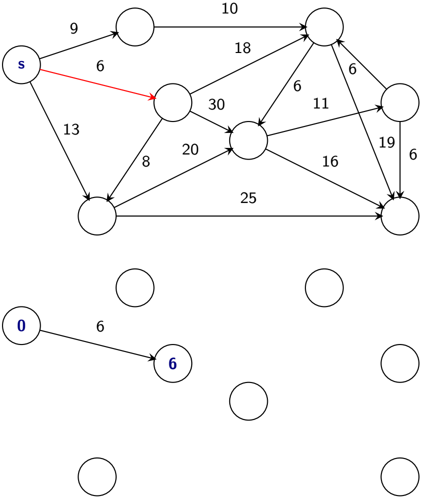

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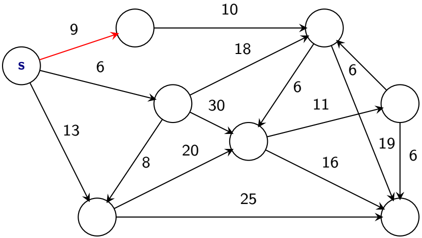

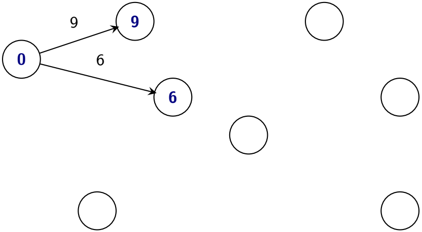

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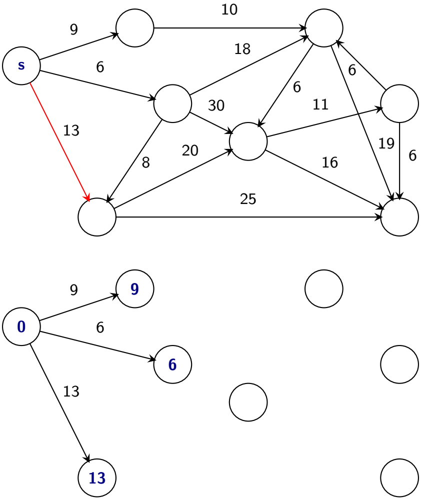

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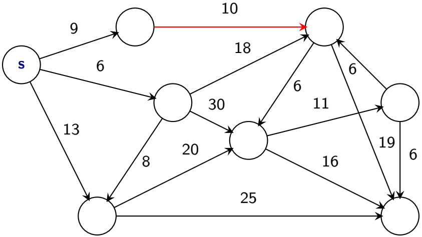

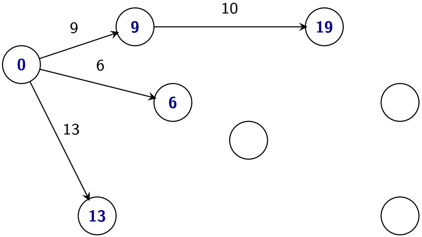

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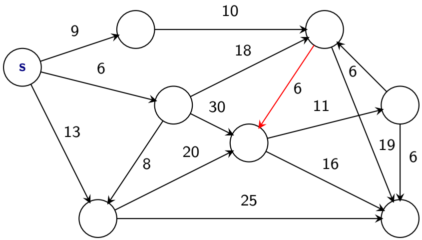

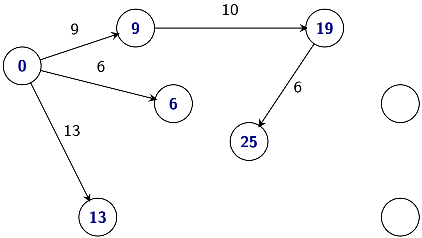

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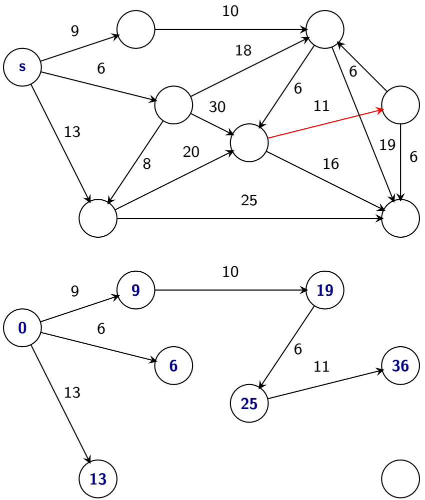

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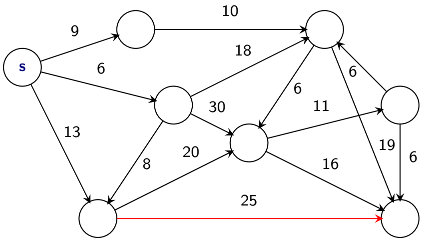

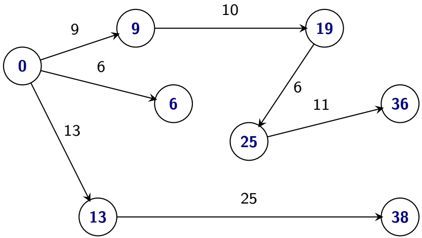

Improved Algorithm

1. Main work is to compute the $\mathbf{d'(s,u)}$ values in each iteration
2. $\mathbf{d'(s,u)}$ changes from iteration $\mathbf{i}$ to $\mathbf{i+1}$ only because of the node $\mathbf{v}$ that is added to $\mathbf{X}$ in iteration $\mathbf{i}$.

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Initialize X = ∅, d'(s, s) = 0
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    Update d'(s, u) for each u in V − X as follows:
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Running time:

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```

Running time: $O(m + n^2)$ time.

1. n outer iterations and in each iteration following steps
2. updating $d'(s, u)$ after v is added takes $O(deg(v))$ time so total work is $O(m)$ since a node enters X only once
3. Finding v from $d'(s, u)$ values is $O(n)$ time

Dijkstra's Algorithm

1. eliminate $\mathbf{d'(s,u)}$ and let $\mathrm{dist}(\mathbf{s},\mathbf{u})$ maintain it
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Priority Queues to maintain **dist** values for faster running time

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Priority Queues to maintain **dist** values for faster running time

1. Using heaps and standard priority queues: $\mathbf{O((m+n)\log n)}$
2. Using Fibonacci heaps: $\mathbf{O(m+n\log n)}$.

Priority Queues

Data structure to store a set $\mathbf{S}$ of $\mathbf{n}$ elements where each element $\mathbf{v} \in \mathbf{S}$ has an associated real/integer key $\mathbf{k(v)}$ such that the following operations:

1. **makePQ**: create an empty queue.
2. **findMin**: find the minimum key in $\mathbf{S}$.
3. **extractMin**: Remove $\mathbf{v} \in \mathbf{S}$ with smallest key and return it.
4. **insert**($\mathbf{v}, \mathbf{k(v)}$): Add new element $\mathbf{v}$ with key $\mathbf{k(v)}$ to $\mathbf{S}$.
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5. **delete**$(\mathbf{v})$: Remove element $\mathbf{v}$ from $\mathbf{S}$.
6. **decreaseKey**$(\mathbf{v}, \mathbf{k'(v)})$: *decrease* key of $\mathbf{v}$ from $\mathbf{k(v)}$ (current key) to $\mathbf{k'(v)}$ (new key). Assumption: $\mathbf{k'(v)} \leq \mathbf{k(v)}$.
7. **meld**: merge two separate priority queues into one.

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All operations can be performed in $\mathbf{O(\log n)}$ time.
decreaseKey is implemented via **delete** and **insert**.

Dijkstra's Algorithm using Priority Queues

```
Q ← makePQ()
insert(Q, (s, 0))
for each node u ≠ s do
    insert(Q, (u, ∞))
X ← ∅
for i = 1 to |V| do
    (v, dist(s, v)) = extractMin(Q)
    X = X ∪ {v}
    for each u in Adj(v) do
        decreaseKey(Q, (u, min(dist(s, u), dist(s, v) + ℓ(v, u)))).
```

Priority Queue operations:

1. $O(n)$ **insert** operations
2. $O(n)$ **extractMin** operations
3. $O(m)$ **decreaseKey** operations

Implementing Priority Queues via Heaps

Using Heaps

Store elements in a heap based on the key value

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Dijkstra's algorithm can be implemented in $\mathbf{O((n + m) \log n)}$ time.

Priority Queues: Fibonacci Heaps/Relaxed Heaps

Fibonacci Heaps

1. **extractMin**, **insert**, **delete**, **meld** in **$O(\log n)$** time
2. **decreaseKey** in **$O(1)$** *amortized* time:

Priority Queues: Fibonacci Heaps/Relaxed Heaps

Fibonacci Heaps

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3. Relaxed Heaps: **decreaseKey** in $\mathbf{O(1)}$ worst case time but at the expense of **meld** (not necessary for Dijkstra's algorithm)

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1. Dijkstra's algorithm can be implemented in $\mathbf{O(n \log n + m)}$ time. If $\mathbf{m = \Omega(n \log n)}$, running time is linear in input size.

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1. Dijkstra's algorithm can be implemented in $\mathbf{O(n \log n + m)}$ time. If $\mathbf{m = \Omega(n \log n)}$, running time is linear in input size.
2. Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps (European Symposium on Algorithms, September 2009!)

Shortest Path Tree

Dijkstra's algorithm finds the shortest path distances from s to $\mathbf{V}$.
Question: How do we find the paths themselves?

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Question: How do we find the paths themselves?

```
Q = makePQ()
insert(Q, (s,0))
prev(s) ← null
for each node u ≠ s do
    insert(Q, (u,∞) )
    prev(u) ← null

X = ∅
for i = 1 to |V| do
    (v, dist(s,v)) = extractMin(Q)
    X = X ∪ {v}
    for each u in Adj(v) do
        if (dist(s,v) + ℓ(v,u) < dist(s,u)) then
            decreaseKey(Q, (u, dist(s,v) + ℓ(v,u)))
            prev(u) = v
```

Shortest Path Tree

Lemma

The edge set $(\mathbf{u}, \mathrm{prev}(\mathbf{u}))$ *is the* reverse *of a shortest path tree rooted at* $\mathbf{s}$*. For each* $\mathbf{u}$*, the reverse of the path from* $\mathbf{u}$ *to* $\mathbf{s}$ *in the tree is a shortest path from* $\mathbf{s}$ *to* $\mathbf{u}$*.*

Proof Sketch.

1. The edge set $\{(\mathbf{u}, \mathrm{prev}(\mathbf{u})) \mid \mathbf{u} \in \mathbf{V}\}$ induces a directed in-tree rooted at $\mathbf{s}$ (Why?)
2. Use induction on $|\mathbf{X}|$ to argue that the tree is a shortest path tree for nodes in $\mathbf{V}$.

$\square$

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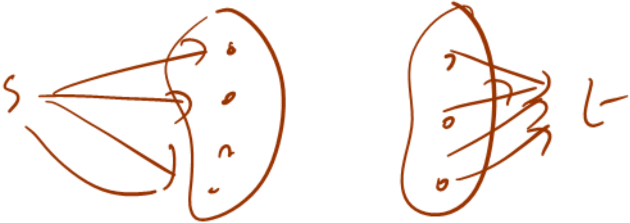
1. In undirected graphs shortest path from s to u is a shortest path from u to s so there is no need to distinguish.
2. In directed graphs, use Dijkstra's algorithm in G^{rev}!

Shortest paths between sets of nodes

Suppose we are given $\mathbf{S} \subset \mathbf{V}$ and $\mathbf{T} \subset \mathbf{V}$. Want to find shortest path from $\mathbf{S}$ to $\mathbf{T}$ defined as:

$$\mathrm{dist}(\mathbf{S}, \mathbf{T}) = \min_{\mathbf{s} \in \mathbf{S}, \mathbf{t} \in \mathbf{T}} \mathrm{dist}(\mathbf{s}, \mathbf{t})$$

How do we find $\mathrm{dist}(\mathbf{S}, \mathbf{T})$?



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Basic solution: Compute for each $\mathbf{x \in X}$, $\mathbf{d(s, x)}$ and $\mathbf{d(x, t)}$ and take minimum. $\mathbf{2|X|}$ shortest path computations. $\mathbf{O(|X|(m + n \log n))}$.

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Better solution: Compute shortest path distances from $\mathbf{s}$ to every node $\mathbf{v \in V}$ with one Dijkstra. Compute from every node $\mathbf{v \in V}$ shortest path distance to $\mathbf{t}$ with one Dijkstra.