CS 374 Lab 25: More NP-Completeness

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Proving that a problem X is NP-hard requires several steps:

- Choose a problem Y that you already know is NP-hard.
- Describe a reduction f from Y to X, i.e., given input w for problem Y, $f(w)$ is an input to problem X.
- Prove that the function f is computable in polynomial time, by outlining an algorithm running in polynomial time that computes f.
- Prove that your reduction f is correct. This almost always requires two separate steps:
  - Prove that if $w \in Y$ then $f(w) \in X$, i.e., the reduction f transforms "yes" instances of Y into "yes" instances of X.
  - Prove that if $w \notin Y$ then $f(w) \notin X$, i.e., the reduction f transforms "no" instances of Y into "no" instances of X. Equivalently: Prove that if $f(w) \in X$ then $w \in Y$.

Proving that X is NP-Complete requires you to **additionally** prove that $X \in NP$ by describing a non-deterministic polynomial-time algorithm for X. Typically this is not hard for the problems we consider but it is not always obvious.

Problem 1. [Category: Proof] A kite is a graph on an even number of nodes, say $2n$, in which n of the nodes form a clique and the remaining n vertices are connected in a "tail" that consists of a path joined to one of the nodes in the clique. Given a graph G and an integer k, the KITE problem asks whether or not there exists a subgraph which is a kite that contains $2k$ nodes. Prove that KITE is NP-Complete.

Problem 2. [Category: Proof] A *Hamiltonian cycle* in an undirected graph G is a cycle that goes through every vertex of G exactly once. The problem of determining whether or not a graph has a Hamiltonian cycle is NP-complete. A **tonian cycle** in an undirected graph G is a cycle that goes through at least *half* of the vertices of G, and a **Hamilhamiltonian circuit** in an undirected graph G is a closed walk that goes through every vertex in G exactly *twice*.

1. Prove that it is NP-hard to determine whether a given graph contains a tonian cycle. (This should be easy: describe a reduction that given a graph G, outputs a graph G' such that G' has a tonian cycle if and only if G has a Hamiltonian cycle.)
2. (harder) Prove that it is NP-hard to determine whether a given graph contains a Hamilhamiltonian circuit. *Hint: your reduction from Hamiltonian cycle should create a graph G' from G by hanging a small "gadget" off of each vertex.*

Problem 3. [Category: Proof] A *Hamiltonian* cycle in a directed or undirected graph G is a cycle that goes through every vertex of G *exactly* once. DirHamiltonian is the problem where you are given a directed graph G and asked to determine if G has a Hamiltonian cycle. Similarly, Hamitonian is the problem where you are asked to determine if an *undirected* graph G has a Hamiltonian cycle.

In this problem, you will show that DIRHAMILTONIAN $\leq_P$ HAMILTONIAN as follows. Given directed graph G form undirected graph G' as follows: G' contains all vertices of G, but in addition, for each vertex v in G, two new vertices are added to G': v^{in} and v^{out}. Edges of G' are

- For each vertex v, undirected edges (v^{in}, v) and (v, v^{out}), are included in G'.
- For each directed edge (u, v) of G, the undirected edge (u^{out}, v^{in}) is included in G'.

1. First draw a simple directed graph G with vertices u, v, w and directed edges $(u, v), (u, w), (v, w)$. Now create G' and check it against a different group's answer to make sure you understand the reduction.
2. Prove the correctness of the reduction.