7 (100 pts.) Fooling sets revisited.

(You can not use the Myhill-Nerode theorem in solving this exercise, since this exercise is the MN theorem.) Let $L$ be a regular language over $\Sigma = \{0, 1\}$. Let $F = \{f_1, \ldots, f_k\}$ be a maximum cardinality fooling set for $L$ ($F$ must be finite, as otherwise $L$ would not be a regular language).

7.A. (40 pts.) Prove that for any string $x \in \Sigma^*$, there exists a unique string $\alpha(x) \in F$, such that $x$ and $\alpha(x)$ are indistinguishable for $L$.

7.B. (30 pts.) Prove that for any string $x \in \Sigma^*$, and $c \in \Sigma$, we have $\alpha(xc) = \alpha(\alpha(x)c)$.

7.C. (30 pts.) Consider the following DFA: $M = (F, \Sigma, \delta, s, A)$, where

\[ \forall f \in F, c \in \Sigma \quad \delta(f, c) = \alpha(fc), \]

$s = \alpha(\varepsilon)$, and $A = F \cap L$.

Prove that $L(M) = L$.

[Hint: First prove that for any $x \in \Sigma^*$, we have $\delta^*(s, x) = \alpha(x)$.]

8 (100 pts.) Context is everything.

Give a context-free grammar (CFG) for each of the following languages. You must provide explanation for how your grammar works, by describing in English what is generated by each non-terminal. (Formal proofs of correctness are not required.)

8.A. (30 pts.) $L_1 = \{0^i1^j0^k \mid i = j + k \text{ and } i, j, k \geq 0\}$.

8.B. (30 pts.) $L_2 = \{x(110)^n x(111)^n \mid x \in \{0, 1\}, n \geq 1\}$.

8.C. (40 pts.) $L_3 = \{1^i0^j1^k0^\ell \mid i + j = k + \ell, \ i, j, k, \ell \geq 0\}$. 