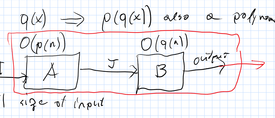
374 Lecture 24 — 11/15/22

Polynomial time reductions

polynomial

$$p(x) = \sum_{i=0}^{n} a_i x^i = O(x^n)$$

$p(x)$ $q(x)$ $\Rightarrow$ $p(q(x))$ also a polynomial



$|I|$ size of input

$|J| = O(p(|I|))$ $\quad n = |I|$

$O(q(|J|))$ + $O(p(n) + q(p(n)))$ ← polynomial

$$p(O(x)) = O(p(x))$$

reduction

X, Y two problems.

R is a (poly) time reduction $\equiv$ R is an algorithm that transforms an instance of X to an instance of Y, such that

I_X is a yes instance $\Leftrightarrow$ $I_Y = R(I_X)$ is a yes instance of Y

instance of X

Clique (problem)

Instance: G graph (undirected), k an integer $\langle G, k\rangle$

Q: Does G contains a clique of size k?

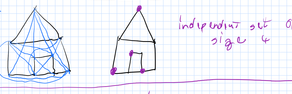
Independent set

Input: $\langle G, k\rangle$

Q: Does G contain an independent set of size k?

$S \subseteq V(G)$ is independent if

$\forall\, u,v \in S \quad uv \notin E(G)$



independent set of size 4

Clique $\leq_p$ Independent set

There is a polynomial time reduction from clique to IS.

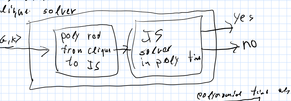
Reduction algorithm (G, k): clique to IS

— $\bar{G} = (V(G), \{uv \mid u,v \in V(G),\ u \neq v,\ uv \notin E(G)\})$ $\quad O(n^2)$

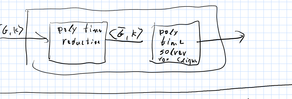
return $\langle \bar{G}, k\rangle$

Lemma G has a clique of size k $\Leftrightarrow$ $\bar{G}$ has an IS of size k.

poly time Clique solver



poly solver for IS gives a polynomial time alg poly solver for clique



$X \leq_p Y$ "reduce X to Y"

$\Rightarrow$ if one can solve Y efficiently then one can solve X efficiently.

$\Rightarrow$ If one can not solve X efficiently then Y can not solve Y efficiently. !!!

Vertex Cover

$G = (V, E)$

vertex cover is a subset of $S \subseteq V$ s.t. all the edges are covered. (guarded)

S is a vertex cover

$\Leftrightarrow$ $uv \in E \quad \{u,v\} \cap S \neq \emptyset$.

VC problem

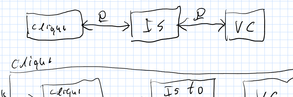
Input: $\langle G, k\rangle$

Q: Is there a vertex cover of size k (VC of size $\leq k$)

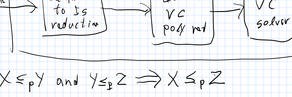
Observation: S is a vertex cover of G

$\Leftrightarrow$ $\bar{S} = V(G) \setminus S$ is an independent set in G.

$VC \leq_p IS$ $\langle G, k\rangle$: return $\langle G, n-k\rangle$ $\qquad$ $IS \leq VC$ $\langle G, k\rangle$: return $\langle G, n-k\rangle$



Clique



$X \leq_p Y$ and $Y \leq_p Z \Rightarrow X \leq_p Z$

Satisfiability

$f(x_1, x_2, \ldots x_n)$ boolean function

Q: Does this function has a satisfying assignment?

x_1	x_2	...	x_n	$f(x_1, x_2, \ldots x_n)$
0	0	0	0	1
				0
				0
				0
				0
				0

x_1	x_2	x_3	Ω	$(x_1=0) \wedge (x_2=1) \wedge (x_3=0)$
0	1	0	1	0
				0

$\bar{x}_1 x_2 \bar{x}_3$

$$f(x_1, x_2, \ldots x_n) = \bigvee \tilde{x}_1 \wedge \tilde{x}_2 \wedge \ldots \tilde{x}_n \qquad \text{DNF}$$

CNF

x_1	x_2	x_3	$x_1 \vee \bar{x}_2 \vee x_3$
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

CNF formula $\bigwedge_{i=1}^{m} C_i$

C_i is an or of n literals

literal is either a variable or its complement.

$C_i = \bigvee \tilde{x}_i$

3CNF

is a CNF where every clause has exactly 3 literals.

SAT problem

Given a CNF formula is it satisfiable

$O(2^n m)$

3SAT = SAT for 3CNF