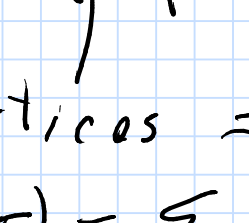


Lecture 23: MST

MST = minimum spanning tree

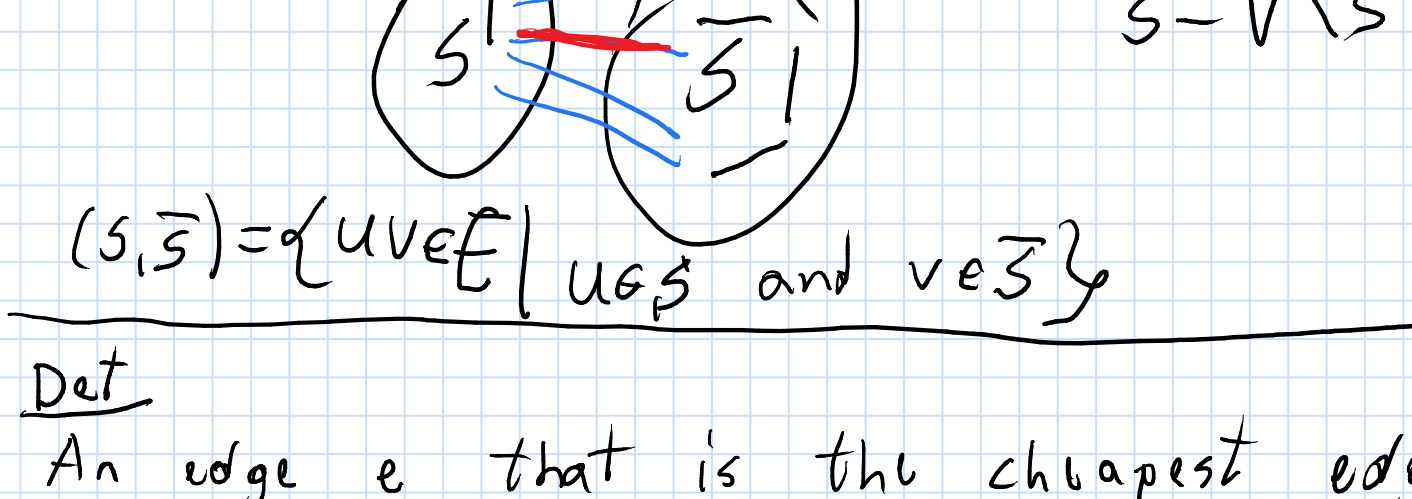
G : Undirected connected graph
 n vertices
 m edges
 distinct weights on the edges (say positive)

Tree - a connected graph with no cycles


 n vertices $\Rightarrow$ tree has $n-1$ edges.
 $cost(T) = \sum_{e \in T} cost(e)$

1926

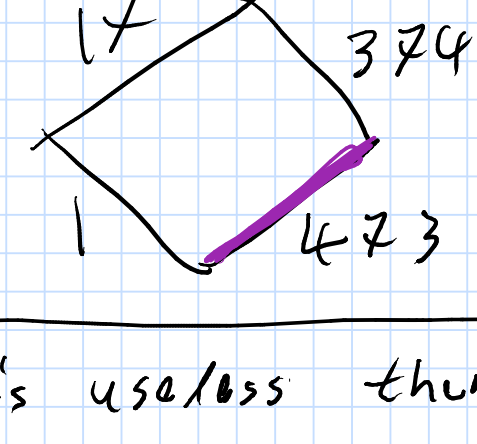
cut $S \subseteq V(G)$ $S \neq \emptyset$ and $S \neq V$



Def: An edge e that is the cheapest edge in any cut is safe.

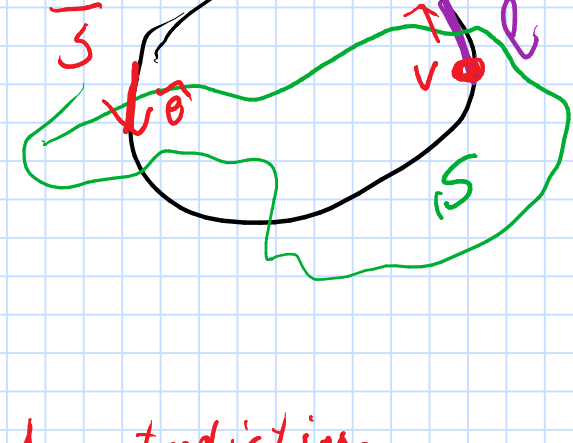
spoiler alert: All the safe edges in the graph form the MST.

An edge e is useless (unsafe) if there exists a cycle C in the graph s.t. e is the most expensive edge in C .



Lemma: If e is useless then it is not safe.

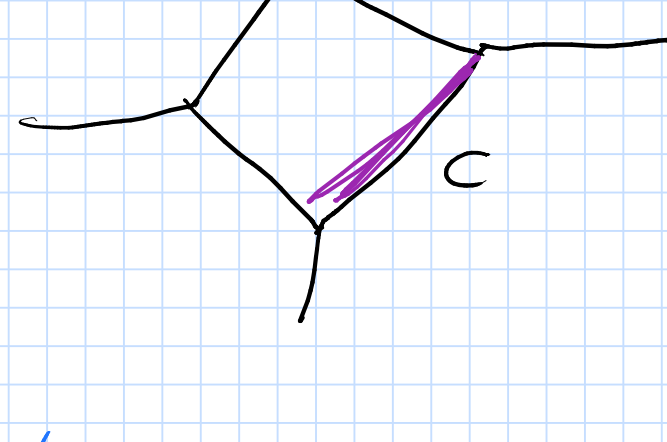
Proof: By contradiction.

$\exists S$ s.t.
 $e = \arg \min_{e' \in (S, \bar{S})} w(e')$

 e is useless $w(e) > w(e')$. A contradiction.
 e is a cheaper edge in the cut defining e .

U : The set of all safe edges in G .

Observations

- U does not contain a cycle.
- U is connected

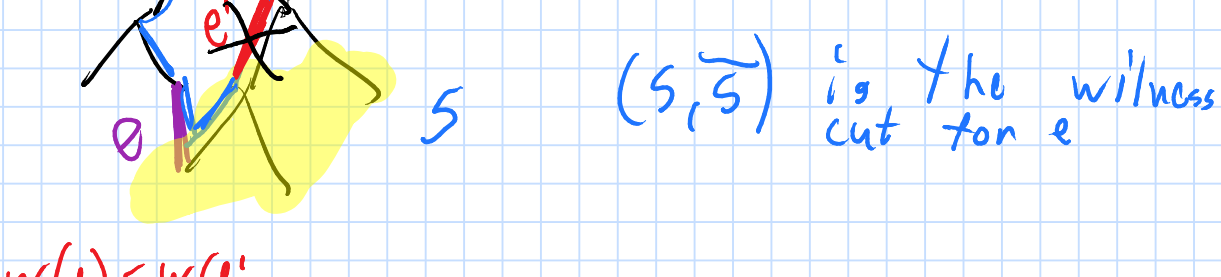


- U spans all the vertices in the graph
 $\Rightarrow U$ forms a spanning tree.

Lemma

Let T be any MST of G , and let e be a safe edge. Then $e \in T$.

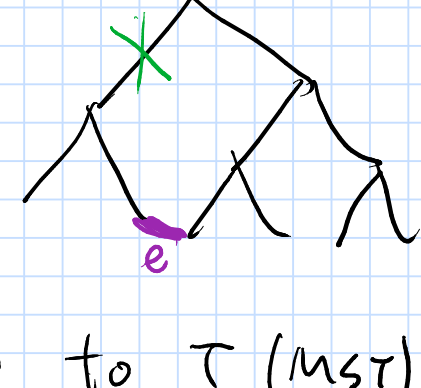
Proof: Assume e not in MST T .



$w(e) < w(e')$
 $T' = (T - e') + e$ $w(T') = w(T) - w(e') + w(e) < w(T)$
 $\Rightarrow T'$ a cheaper spanning tree $\Rightarrow$ a contradiction.

Lemma: If an edge e is not in the MST T then e is useless.

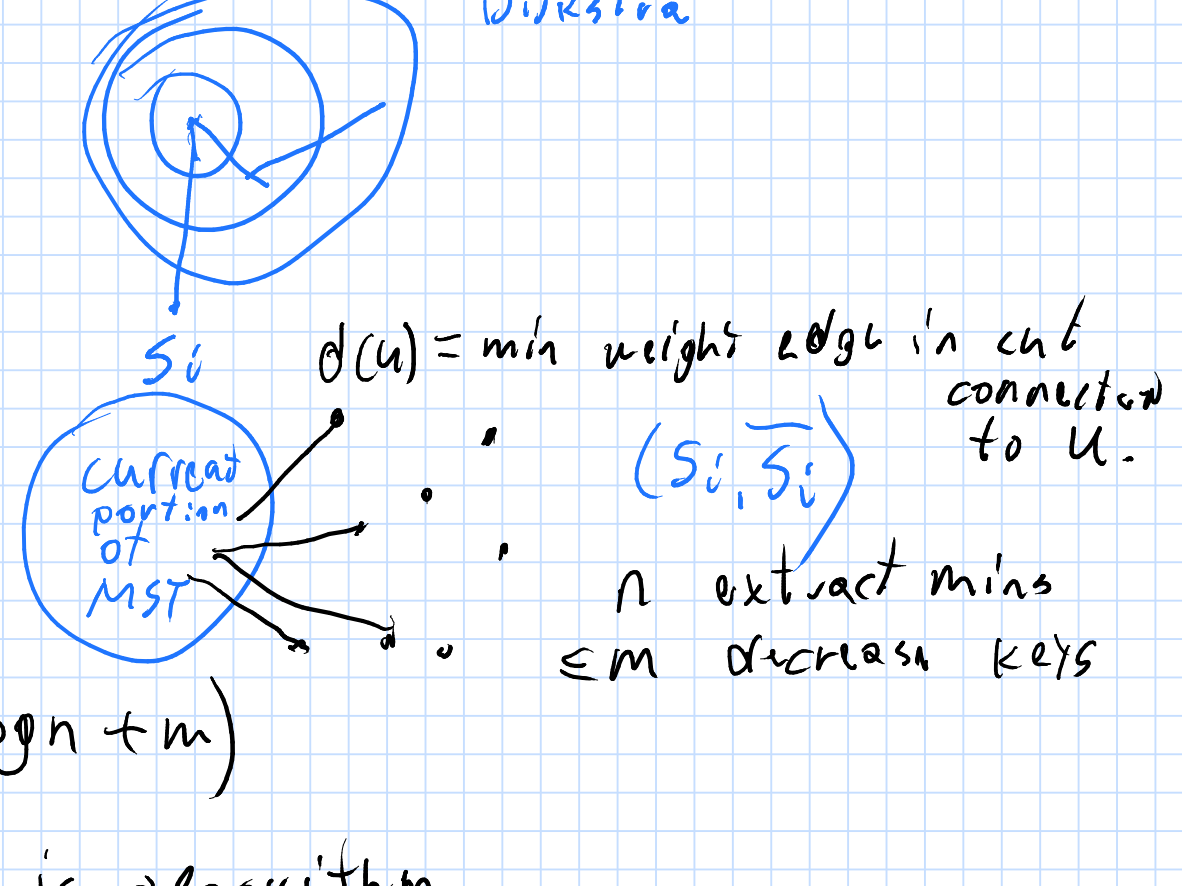
Proof



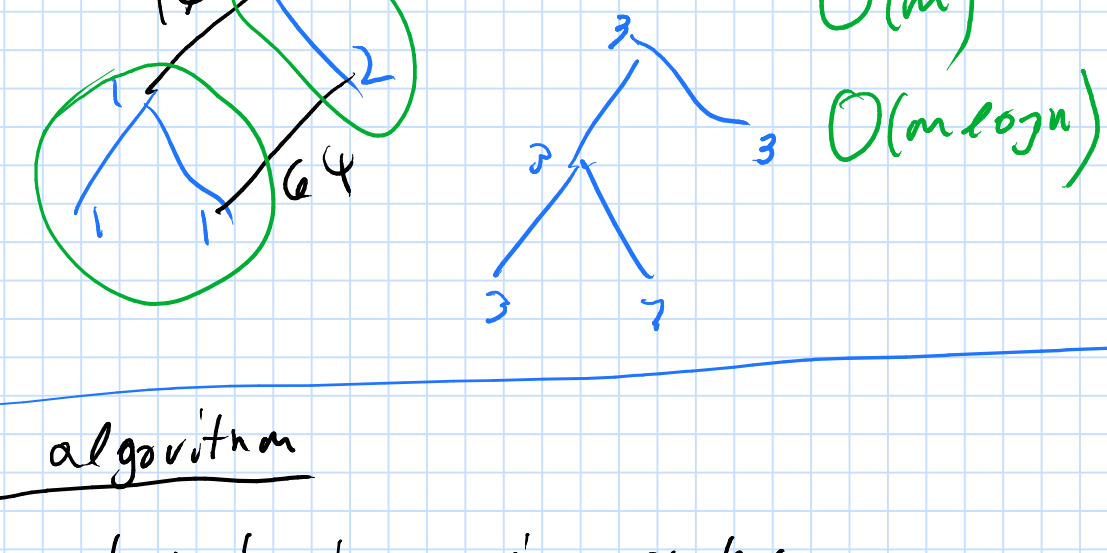
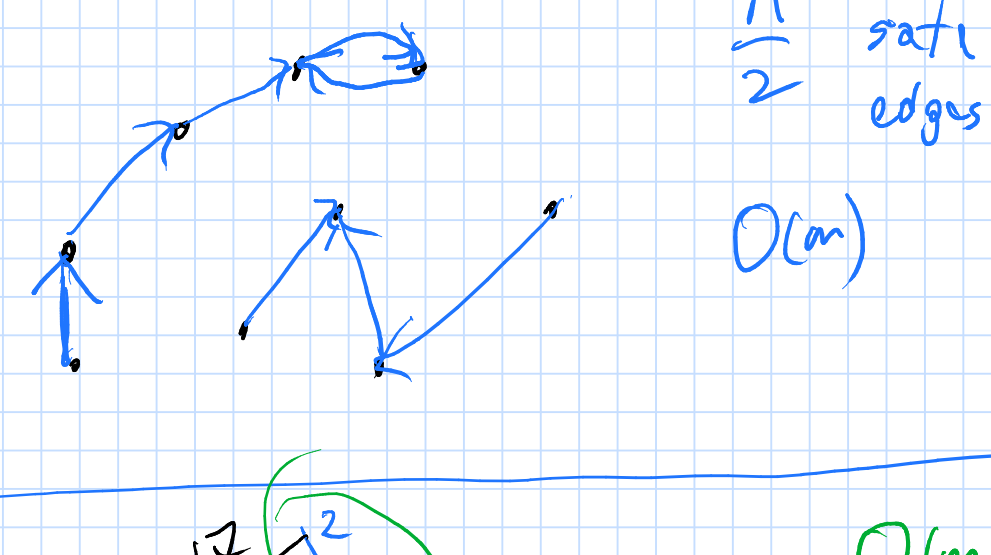
adding e to T (MST) forms a cycle C
 $\Rightarrow e$ must be most expensive edge on C
 as otherwise a beneficial exchange exists and T is not the MST.
 $\Rightarrow e$ is useless.

Algorithms

Prim's algorithm



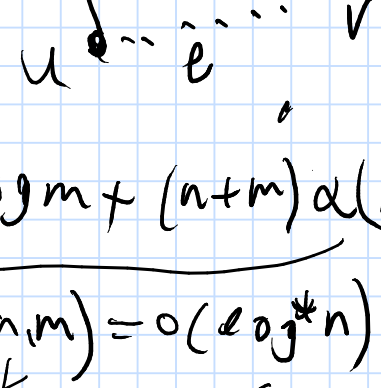
Boruvka's algorithm



Kruskal's algorithm

- Sort the edges in increasing order.

Union-Find



$O(m \log m + (n+m) \alpha(n, m))$
 $\hookrightarrow \alpha(n, m) = o(\log^* n)$

Deletion algorithm

$n \log n + m$ Prim's alg
 only compute MST deterministically in linear time?
 $O(n+m)$ randomized algorithm.

we have an optimal alg to compute MST.