374 Lecture 21 Negative weights and shortest path

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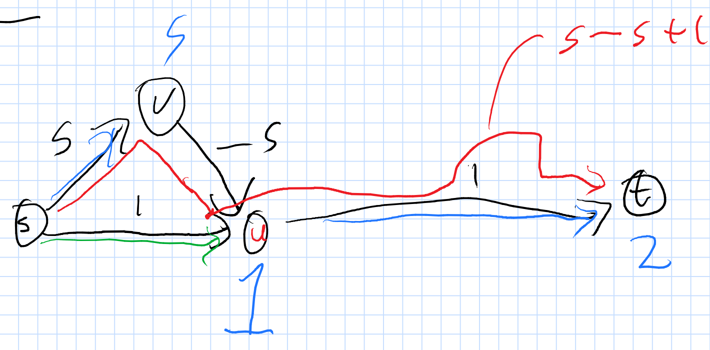
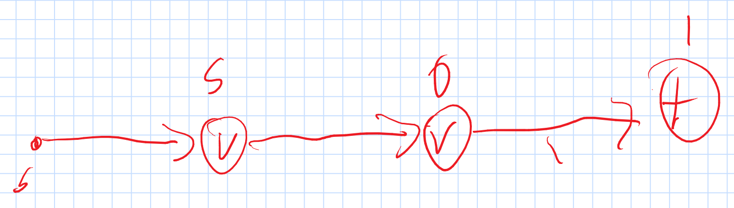
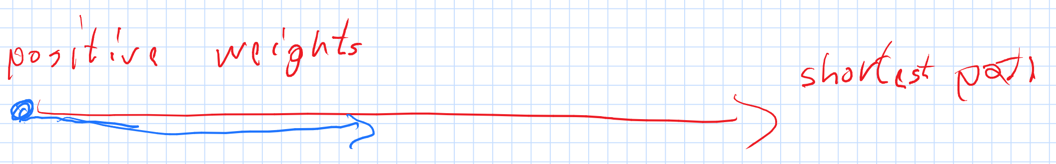
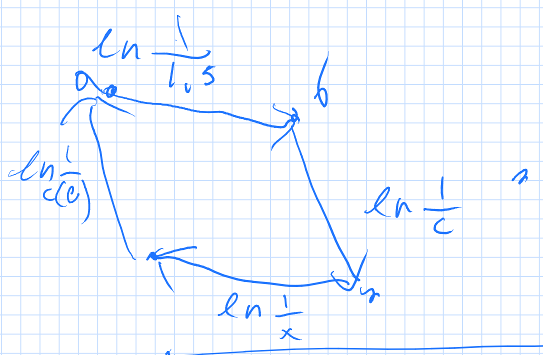
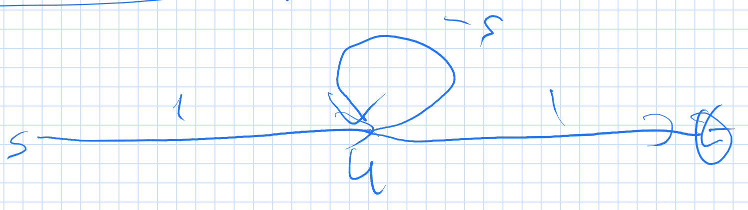








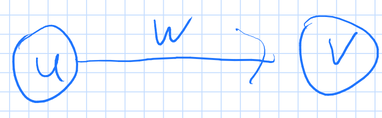


2

$1-5+1=-3$

$1-5\cdot 2+1=-8$

⋮



$d(u)$: curr shortest path from s to u
$d(v)$

$u \to v$ was <u>tence</u> if $d(u)+w(u\to v)<d(v)$

relax $(u\to v)$:
  if $d(v) < d(u)+w(u\to v)$ then
    $d(v) \leftarrow d(u)+w(u\to v)$.

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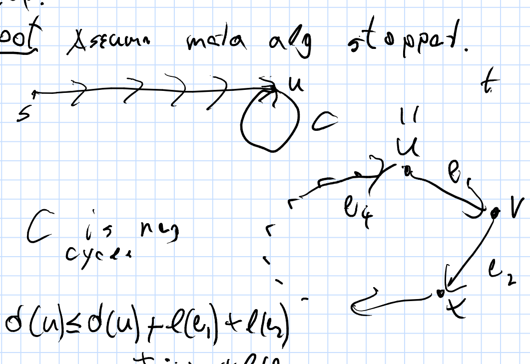
meta-algorithm

$d(s) \leftarrow 0 \quad \forall v \in V \setminus s \quad d(v) \leftarrow +\infty$

as long $\exists e$ tence do
  relax(e)
return $d(t)$.

Claim If G has negative cycle reachable from s then there is always a tense edge and the meta-alg will not stop.

Proof Assume meta alg stopped.



$d(v) \le d(u)+\ell(e_1)$

$d(x) \le d(v)+\ell(e_2)$

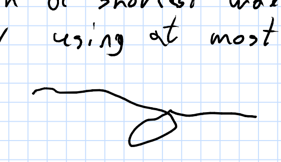
$d(x) \le d(u)+\ell(e_1)+\ell(e_2)$

$d(u) \le d(u)+\ell(e_1)+\ell(e_2)+\dots+\ell(e_k)$

$\le d(u)+$ negative number

$d(v,i) \equiv$ length of shortest walk from s to v using at most i edges.

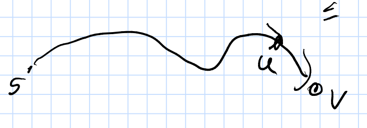
walk



longest path in a graph with positive weights is NPC

$$d(v,0) = \begin{cases} 0 & v=s \\ +\infty & v \ne s \end{cases}$$

$$d(v,i) = \min\left(\min_{(u\to v)\in E(G)} \left(d(u,i-1)+\ell(u\to v)\right),\ d(v,i-1)\right)$$



$\le \quad d(u,i-1)+\ell(u\to v)$

init $d(v,0)$ $\forall v \in V$.
For $i=1$ to $n-1$ do
  for $v \in V$ do
    $d(v,i) \leftarrow d(v,i-1)$
  for $u \to v \in E$ do
    $d(v,i) \leftarrow \min(d(v,i),\ d(u,i-1)+\ell(u,v))$
for $u\to v \in E$ do
  if $d(v,n-1) < d(u,n-1)+\ell(u\to v)$ then
    output "there is a negative cycle reachable from s".

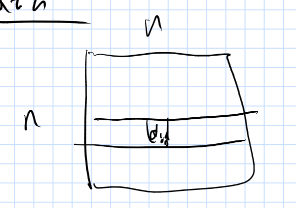
$\boxed{O(n\cdot m)}$

Bellman-Ford $O(nm)=O(n^3)$

Dijkstra $O(n\log n+m)$

All pairs shortest path

All positive weighs
$n \times$ dijkstra.
$O(n^2\log n+nm)$
$=O(n^3)$



negative weights

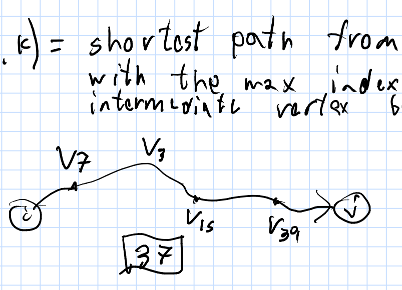
$n \times$ Bellman-Ford $\quad O(n\cdot nm)=O(n^2m)$
$=O(n^4)$.

Floyd-warshall ~~dese~~ all pairs shortest path

$O(n^3)$

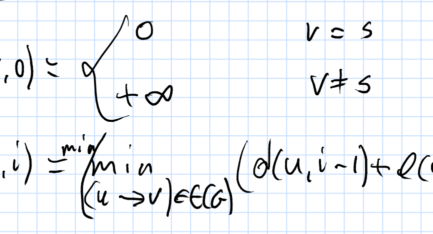
$V_1, V_2, \dots, V_n$

$d(i,j,k)=$ shortest path from V_i to V_j with the max index of any intermediate vertex being k.



$$d(i,j,0) = \begin{cases} \ell(v_i\to v_j) & v_i \to v_j \in E(G) \\ +\infty & \text{otherwise} \end{cases}$$

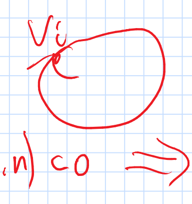
$$d(i,j,k)=\min\begin{cases} d(i,j,k-1) \\ d(i,k,k-1)+d(k,j,k-1) \end{cases}$$



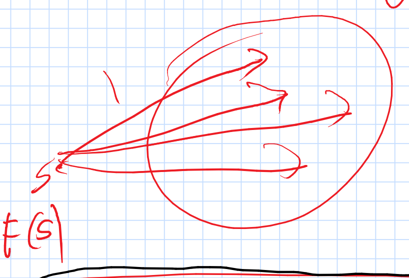
$d(i,j,n)$ $\forall i,j$

$O(n^3)$

FW(G)
  $G=(\{v_1,v_2,\dots,v_n\},E)$
  for $i=1\dots n$ do
    for $j=1\dots n$ do
      $d(i,j,0)=\begin{cases} \ell(v_i\to v_j) & v_i\to v_j \in E \\ +\infty \end{cases}$
  for $k=1$ to n do
    for $i=1$ to n do
      for $j=1$ to n do
        $d(i,j,k)=\min\begin{cases} d(i,j,k-1) \\ d(i,k,k-1)+d(k,j,k-1) \end{cases}$
  output $d(i,j,n)$ $\forall i,j$.

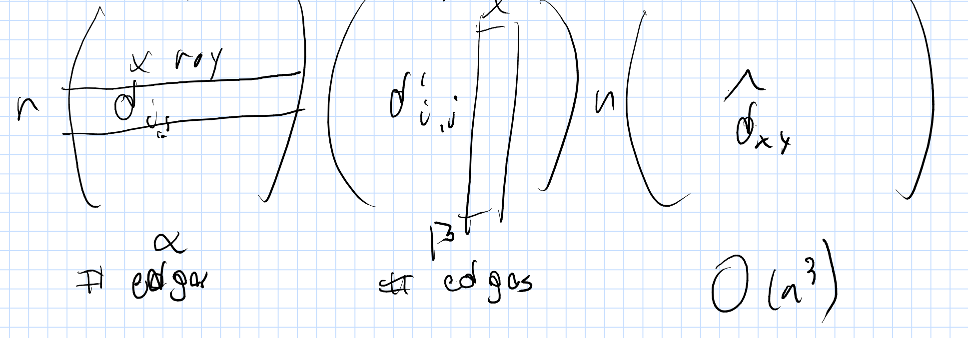


$\exists i \quad d(i,i,n)<0 \Rightarrow \exists$ negative cycle in the graph.



$O(nm) < O(n^3)$

BF(G)



α # edges $\quad$ β # edges

$O(n^3)$
⇓
$O(n^{2.31\dots})$

$\hat{d}_{xy}=\min_{t=1}^{n}\left(d_{x,t}+d'_{t,y}\right)$

$\alpha+\beta$ # edges