# Deterministic Finite Automata (DFAs)

Lecture 3

Tuesday, August 30, 2022

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# 3.1
# DFA Introduction

# DFAs also called Finite State Machines (FSMs)

- The "simplest" model for computers?
- State machines that are common in practice.
  - Vending machines
  - Elevators
  - Digital watches
  - Simple network protocols
- Programs with fixed memory

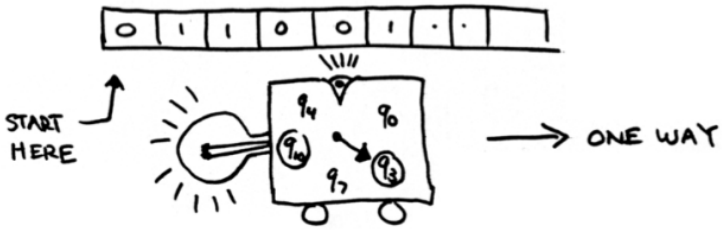
# A simple program

Program to check if a given input string $\mathbf{w}$ has odd length

```
int n = 0
While input is not finished
    read next character c
    n ← n + 1
endWhile
If (n is odd) output YES
Else output NO
```

```
bit x = 0
While input is not finished
    read next character c
    x ← flip(x)
endWhile
If (x = 1) output YES
Else output NO
```

# Another view



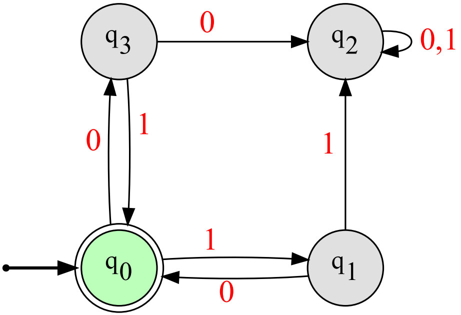

- Machine has input written on a read-only tape
- Start in specified start state
- Start at left, scan symbol, change state and move right
- Circled states are accepting
- Machine accepts input string if it is in an accepting state after scanning the last symbol.

# Draw me a ~~sheep~~ DFA

DFA to check if a given input string has odd length

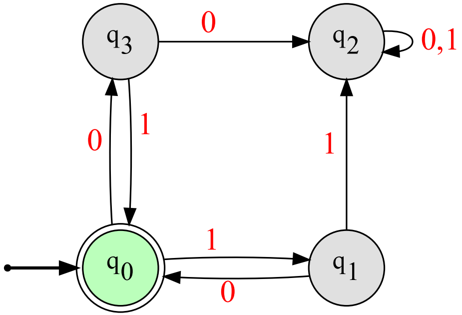
# 3.1.1
# Graphical representation of DFA

# Graphical Representation/State Machine



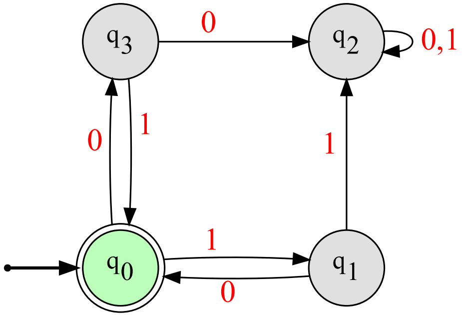

- Directed graph with nodes representing states and edge/arcs representing transitions labeled by symbols in $\Sigma$
- For each state (vertex) $q$ and symbol $a \in \Sigma$ there is exactly one outgoing edge labeled by $a$
- Initial/start state has a pointer (or labeled as $s$, $q_0$ or "start")
- Some states with double circles labeled as accepting/final states

# Graphical Representation




- Where does **001** lead?
- Where does **10010** lead?
- Which strings end up in accepting state?
- Can you prove it?
- Every string **w** has a unique walk that it follows from a given state **q** by reading one letter of **w** from left to right.

# Graphical Representation




**Definition 3.1.**

A DFA **M** accepts a string **w** iff the unique walk starting at the start state and spelling out **w** ends in an accepting state.

**Definition 3.2.**

The language accepted (or recognized) by a DFA **M** is denote by **L(M)** and defined as: $\mathbf{L(M)} = \{\mathbf{w} \mid \mathbf{M} \text{ accepts } \mathbf{w}\}$.

# Warning

"**M** accepts language **L**" does not mean simply that that **M** accepts each string in **L**.

It means that **M** accepts each string in **L** and no others. Equivalently **M** accepts each string in **L** and does not accept/rejects strings in $\Sigma^* \setminus L$.

**M** "recognizes" **L** is a better term but "accepts" is widely accepted (and recognized) (joke attributed to Lenny Pitt)

# 3.1.2
# Formal definition of DFA

# Formal Tuple Notation

## Definition 3.3.

A deterministic finite automata $(\mathrm{DFA})$ $\mathbf{M} = (\mathbf{Q}, \mathbf{\Sigma}, \delta, \mathbf{s}, \mathbf{A})$ is a five tuple where

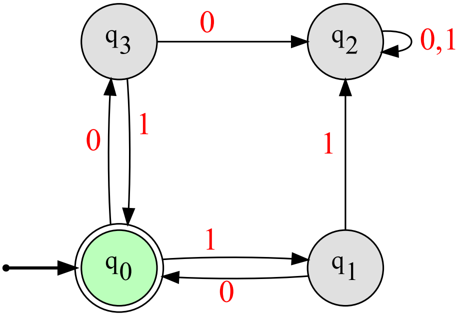
- $\mathbf{Q}$ is a finite set whose elements are called states,
- $\mathbf{\Sigma}$ is a finite set called the input alphabet,
- $\delta : \mathbf{Q} \times \mathbf{\Sigma} \to \mathbf{Q}$ is the transition function,
- $\mathbf{s} \in \mathbf{Q}$ is the start state,
- $\mathbf{A} \subseteq \mathbf{Q}$ is the set of accepting/final states.

Common alternate notation: $\mathbf{q_0}$ for start state, $\mathbf{F}$ for final states.

# DFA Notation

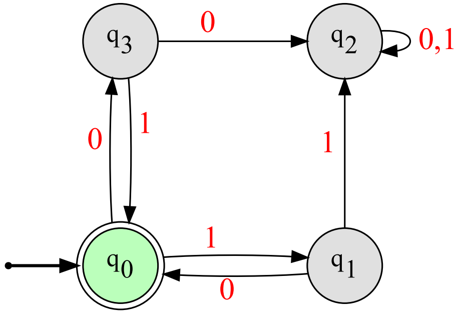
$$\mathbf{M} = \Big( \overbrace{\mathbf{Q}}^{\text{set of all states}}, \underbrace{\mathbf{\Sigma}}_{\text{alphabet}}, \overbrace{\delta}^{\text{transition func}}, \underbrace{\mathbf{s}}_{\text{start state}}, \overbrace{\mathbf{A}}^{\text{set of all accept states}} \Big)$$

# Example




- $\mathbf{Q = \{q_0, q_1, q_1, q_3\}}$
- $\mathbf{\Sigma = \{0, 1\}}$
- $\delta$
- $\mathbf{s = q_0}$
- $\mathbf{A = \{q_0\}}$

# Example: The transition function




$\delta$ :

| state $q$ | input $c$ | result $\delta(q, c)$ |
|---|---|---|
| $\in Q$ | $\in \Sigma$ | $\in Q$ |
| $q_0$ | 0 | $q_3$ |
| $q_0$ | 1 | $q_1$ |
| $q_1$ | 0 | $q_0$ |
| $q_1$ | 1 | $q_2$ |
| $q_2$ | 0 | $q_2$ |
| $q_2$ | 1 | $q_2$ |
| $q_3$ | 0 | $q_2$ |
| $q_3$ | 1 | $q_0$ |

# 3.1.3
# Extending the transition function to strings

# Extending the transition function to strings

Given DFA $\mathbf{M = (Q, \Sigma, \delta, s, A)}$, $\delta(\mathbf{q}, \mathbf{a})$ is the state that $\mathbf{M}$ goes to from $\mathbf{q}$ on reading letter $\mathbf{a}$

Useful to have notation to specify the unique state that $\mathbf{M}$ will reach from $\mathbf{q}$ on reading string $\mathbf{w}$

Transition function $\delta^* : \mathbf{Q} \times \mathbf{\Sigma}^* \rightarrow \mathbf{Q}$ defined inductively as follows:

- $\delta^*(\mathbf{q}, \mathbf{w}) = \mathbf{q}$ if $\mathbf{w} = \epsilon$
- $\delta^*(\mathbf{q}, \mathbf{w}) = \delta^*(\delta(\mathbf{q}, \mathbf{a}), \mathbf{x})$ if $\mathbf{w} = \mathbf{ax}$.

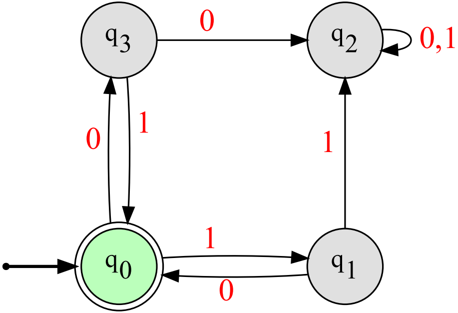
# Formal definition of language accepted by $\mathbf{M}$

**Definition 3.4.**

The language $\mathbf{L(M)}$ accepted by a DFA $\mathbf{M = (Q, \Sigma, \delta, s, A)}$ is

$$\{\mathbf{w} \in \mathbf{\Sigma}^* \mid \delta^*(\mathbf{s}, \mathbf{w}) \in \mathbf{A}\}.$$

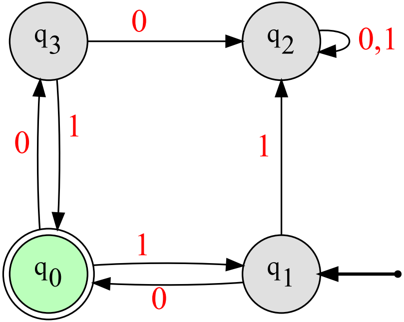
# Example




What is:

- $\boldsymbol{\delta^*(q_1, \epsilon)}$
- $\boldsymbol{\delta^*(q_0, 1011)}$
- $\boldsymbol{\delta^*(q_1, 010)}$
- $\boldsymbol{\delta^*(q_4, 10)}$
- So what is **L(M)**??????

# Example continued

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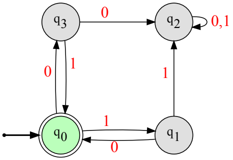
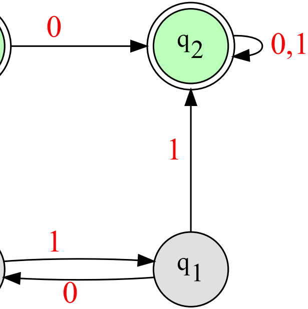



- What is **L(M)** if start state is changed to $\mathbf{q_1}$?

# Example continued

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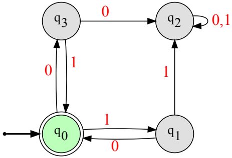
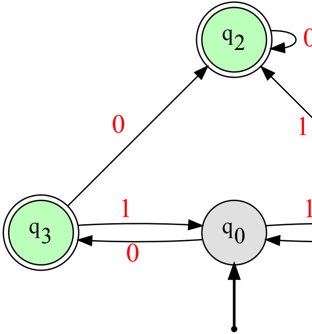



Redraw:




- What is $\mathbf{L(M)}$ if final/accept states are set to $\{\mathbf{q_2}, \mathbf{q_3}\}$ instead of $\{\mathbf{q_0}\}$?

# Advantages of formal specification

- Necessary for proofs
- Necessary to specify abstractly for class of languages

**Exercise:** Prove by induction that for any two strings $\mathbf{u, v}$, any state $\mathbf{q}$, $\delta^*(\mathbf{q}, \mathbf{uv}) = \delta^*(\delta^*(\mathbf{q}, \mathbf{u}), \mathbf{v})$.

# 3.2
# Constructing DFAs

# DFAs: State = Memory

How do we design a DFA **M** for a given language **L**? That is **L(M) = L**.

- DFA is a like a program that has fixed amount of memory independent of input size.
- The memory of a DFA is encoded in its states
- The state/memory must capture enough information from the input seen so far that it is sufficient for the suffix that is yet to be seen (note that DFA cannot go back)

# DFA Construction: Examples

## Example I: Basic languages

Assume $\boldsymbol{\Sigma = \{0, 1\}}$.

$\mathbf{L} = \emptyset$, $\mathbf{L} = \boldsymbol{\Sigma}^*$, $\mathbf{L} = \{\epsilon\}$, $\mathbf{L} = \{\mathbf{0}\}$.

# DFA Construction: Examples

## Example II: Length divisible by 5

Assume $\mathbf{\Sigma = \{0, 1\}}$.

$\mathbf{L = \{w \in \{0, 1\}^* \mid |w| \text{ is divisible by } 5\}}$

# DFA Construction: examples

## Example III: Ends with 01

Assume $\boldsymbol{\Sigma = \{0, 1\}}$.

$\mathbf{L = \{w \in \{0, 1\}^* \mid w \text{ ends with } 01\}}$

# DFA Construction: examples

## Example IV: Contains 001

Assume $\mathbf{\Sigma = \{0, 1\}}$.
$\mathbf{L} = \{\mathbf{w} \in \{0, 1\}^* \mid \mathbf{w} \text{ contains 001 as substring}\}$

# DFA Construction: examples

## Example V: Contains 001 or 010

Assume $\mathbf{\Sigma = \{0, 1\}}$.

$\mathbf{L = \{w \in \{0,1\}^* \mid w \text{ contains 001 or 010 as substring}\}}$

# DFA construction examples

## Example VI: Has a **1** exactly **k** positions from end

Assume $\Sigma = \{0, 1\}$.

$L = \{w \mid w \text{ has a } 1 \ k \text{ positions from the end}\}$.

# DFA Construction: Example

$\mathbf{L} = \{$Binary numbers congruent to $\mathbf{0 \mod 5}\}$
Example:

1. $\mathbf{1101011_2 = 107_{10} = 2 \mod 5}$,
2. $\mathbf{1010_2 = 10 = 0 \mod 5}$

**Key observation:**
$\mathbf{val(w) \mod 5 = a}$ implies

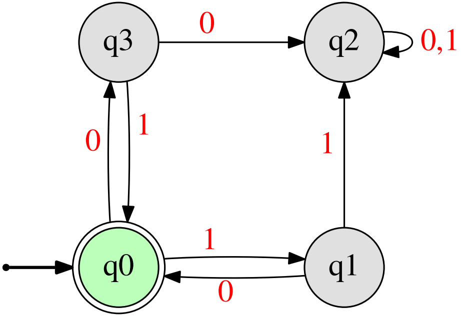
$$\mathbf{val(w0) \mod 5 = (val(w) * 2) \mod 5 = 2a \mod 5}$$

$$\mathbf{val(w1) \mod 5 = \big(val(w) \cdot 2 + 1\big) \mod 5 = (2a + 1) \mod 5}$$

# 3.3
# Complement language

# Complement

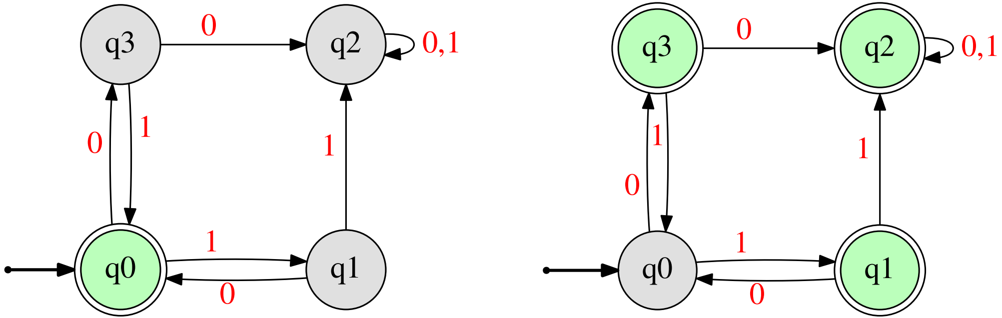
**Question:** If $\mathbf{M}$ is a DFA, is there a DFA $\mathbf{M'}$ such that $\mathbf{L(M') = \Sigma^* \setminus L(M)}$? That is, are languages recognized by DFAs closed under complement?



# Complement

## Example...

Just flip the state of the states!



# Complement

**Theorem 3.1.**

*Languages accepted by* DFA*s are closed under complement.*

Proof.

Let $\mathbf{M} = (\mathbf{Q}, \boldsymbol{\Sigma}, \delta, \mathbf{s}, \mathbf{A})$ such that $\mathbf{L} = \mathbf{L}(\mathbf{M})$.
Let $\mathbf{M}' = (\mathbf{Q}, \boldsymbol{\Sigma}, \delta, \mathbf{s}, \mathbf{Q} \setminus \mathbf{A})$. Claim: $\mathbf{L}(\mathbf{M}') = \bar{\mathbf{L}}$. Why?
$\delta^*_{\mathbf{M}} = \delta^*_{\mathbf{M}'}$. Thus, for every string $\mathbf{w}$, $\delta^*_{\mathbf{M}}(\mathbf{s}, \mathbf{w}) = \delta^*_{\mathbf{M}'}(\mathbf{s}, \mathbf{w})$.
$\delta^*_{\mathbf{M}}(\mathbf{s}, \mathbf{w}) \in \mathbf{A} \Rightarrow \delta^*_{\mathbf{M}'}(\mathbf{s}, \mathbf{w}) \notin \mathbf{Q} \setminus \mathbf{A}$. $\delta^*_{\mathbf{M}}(\mathbf{s}, \mathbf{w}) \notin \mathbf{A} \Rightarrow \delta^*_{\mathbf{M}'}(\mathbf{s}, \mathbf{w}) \in \mathbf{Q} \setminus \mathbf{A}$. □

# 3.4
# Product Construction

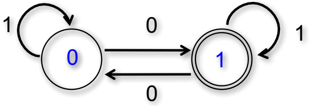
# Union and Intersection

**Question:** Are languages accepted by DFAs closed under union? That is, given DFAs $\mathbf{M_1}$ and $\mathbf{M_2}$ is there a DFA that accepts $\mathbf{L(M_1) \cup L(M_2)}$?
How about intersection $\mathbf{L(M_1) \cap L(M_2)}$?

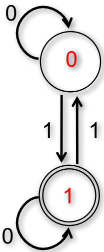
Idea from programming: on input string $\mathbf{w}$

- Simulate $\mathbf{M_1}$ on $\mathbf{w}$
- Simulate $\mathbf{M_2}$ on $\mathbf{w}$
- If both accept than $\mathbf{w} \in \mathbf{L(M_1) \cap L(M_2)}$. If at least one accepts then $\mathbf{w} \in \mathbf{L(M_1) \cup L(M_2)}$.
- Catch: We want a single DFA $\mathbf{M}$ that can only read $\mathbf{w}$ once.
- Solution: Simulate $\mathbf{M_1}$ and $\mathbf{M_2}$ in parallel by keeping track of states of both machines

# Example

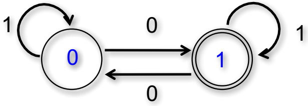



$M_1$ accepts #0 = odd

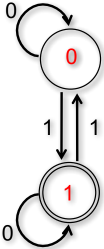



$M_2$ accepts #1 = odd

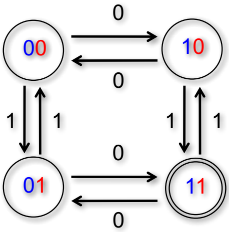
# Example




$M_1$ accepts #0 = odd



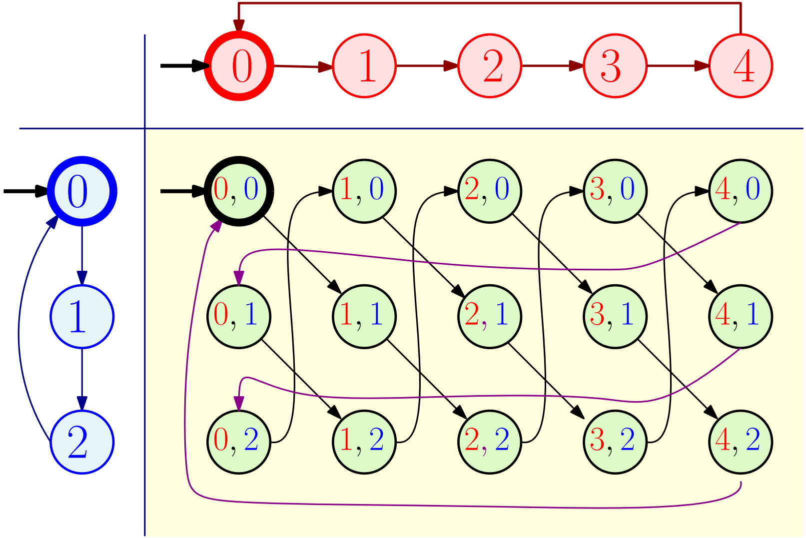

$M_2$ accepts #1 = odd




*Cross-product machine*

# Example II

Accept all binary strings of length divisible by **3** and **5**




Assume all edges are labeled by **0**,**1**.

# Product construction for intersection

$M_1 = (Q_1, \Sigma, \delta_1, s_1, A_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, s_2, A_2)$

Create $M = (Q, \Sigma, \delta, s, A)$ where

- $Q = Q_1 \times Q_2 = \{(q_1, q_2) \mid q_1 \in Q_1, q_2 \in Q_2\}$
- $s = (s_1, s_2)$
- $\delta : Q \times \Sigma \to Q$ where

$$\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$$

- $A = A_1 \times A_2 = \{(q_1, q_2) \mid q_1 \in A_1, q_2 \in A_2\}$

**Theorem 3.1.**

$L(M) = L(M_1) \cap L(M_2)$.

# Correctness of construction

**Lemma 3.2.**

*For each string* $\mathbf{w}$, $\delta^*(\mathbf{s}, \mathbf{w}) = (\delta_1^*(\mathbf{s}_1, \mathbf{w}), \delta_2^*(\mathbf{s}_2, \mathbf{w}))$.

**Exercise:** Assuming lemma prove the theorem in previous slide. Proof of lemma by induction on $|\mathbf{w}|$

# Product construction for union

$\mathbf{M_1 = (Q_1, \Sigma, \delta_1, s_1, A_1)}$ and $\mathbf{M_2 = (Q_1, \Sigma, \delta_2, s_2, A_2)}$

Create $\mathbf{M = (Q, \Sigma, \delta, s, A)}$ where

- $\mathbf{Q = Q_1 \times Q_2 = \{(q_1, q_2) \mid q_1 \in Q_1, q_2 \in Q_2\}}$
- $\mathbf{s = (s_1, s_2)}$
- $\mathbf{\delta : Q \times \Sigma \to Q}$ where

$$\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$$

- $\mathbf{A = \{(q_1, q_2) \mid q_1 \in A_1}$ or $\mathbf{q_2 \in A_2\}}$

**Theorem 3.3.**

$\mathbf{L(M) = L(M_1) \cup L(M_2)}$.

# Set Difference

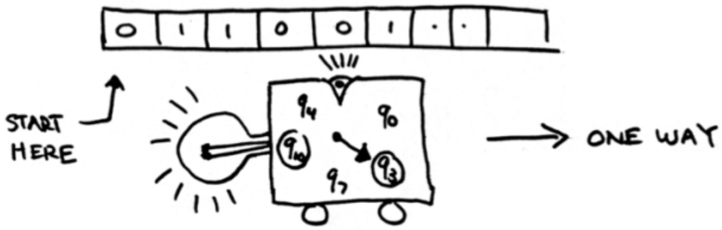
**Theorem 3.4.**

$\mathbf{M_1}, \mathbf{M_2}$ DFAs. *There is a* DFA $\mathbf{M}$ *such that* $\mathbf{L(M)} = \mathbf{L(M_1)} \setminus \mathbf{L(M_2)}$.

**Exercise:** Prove the above using two methods.

- Using a direct product construction
- Using closure under complement and intersection and union

# Things to know: 2-way DFA




**Question:** Why are DFAs required to only move right?
Can we allow DFA to scan back and forth? Caveat: Tape is read-only so only memory is in machine's state.

- Can define a formal notion of a "2-way" DFA
- Can show that any language recognized by a 2-way DFA can be recognized by a regular (1-way) DFA
- Proof is tricky simulation via NFAs

# 3.5
# Supplemental: DFA philosophy

# A finite program can be simulated by a DFA...

1. Finite program = a program that uses a prespecified bounded amount of memory.
2. Given DFA and input, easy to decide if DFA accepts input.
3. A finite program is a DFA!
   # of states of memory of a finite program = finite.
   **# states $\approx$ $\mathbf{2}^{\#\text{ of memory bits used by program}}$**
4. Program using **1K** memory = has...
5. Turing halting theorem: Not possible (in general) to decide if a program stops on an input.
6. DFA $\neq$ programs.

# But universe is finite...

1. Estimate # of atoms in the universe is $\mathbf{10^{82}}$.
2. Assuming each atom can store only finite number of bits.
3. So... number of states of the universe is finite!
4. So... All programs in this universe are DFAs.
5. Checkmate Mate!
6. What is all this nonsense?

# So what is going on...

1. Theory models the world. (Oversimplifies it.)
2. Make it possible to think about it.
3. There are cases where theory does not model the world well.
4. Know when to apply the theory.
5. Reject statements that are correct but not useful.
6. Really Large finite numbers are