Strings and Languages

Lecture 1

Tuesday, August 23, 2022

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1.1
Strings

Alphabet

An alphabet is a **finite** set of symbols.

Examples of alphabets:

- $\mathbf{\Sigma = \{0, 1\}}$,
- $\mathbf{\Sigma = \{a, b, c, \ldots, z\}}$,
- ASCII.
- UTF8.
- $\mathbf{\Sigma = \{\langle moveforward \rangle, \ \langle moveback \rangle\}}$

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String Definitions

Definition 1.1.

1. A string/word over Σ is a **finite sequence** of symbols over Σ. For example, '$\mathbf{0101001}$', '$\mathbf{string}$', '$\langle\mathbf{moveback}\rangle\langle\mathbf{rotate90}\rangle$'
2. ϵ is the empty string.
3. The length of a string $\mathbf{w}$ (denoted by $|\mathbf{w}|$) is the number of symbols in $\mathbf{w}$. For example, $|\mathbf{101}| = \mathbf{3}$, $|\epsilon| = \mathbf{0}$
4. For integer $\mathbf{n} \geq \mathbf{0}$, $\Sigma^{\mathbf{n}}$ is set of all strings over Σ of length $\mathbf{n}$. Σ^* is the set of all strings over Σ.

Inductive/recursive definition of strings

Formal definition of a **string**:

- ϵ is a string of length $\mathbf{0}$
- $\mathbf{ax}$ is a string if $\mathbf{a} \in \mathbf{\Sigma}$ and $\mathbf{x}$ is a string. The length of $\mathbf{ax}$ is $\mathbf{1} + |\mathbf{x}|$

The above definition helps prove statements rigorously via induction.

- Alternative recursive definition useful in some proofs: $\mathbf{xa}$ is a string if $\mathbf{a} \in \mathbf{\Sigma}$ and $\mathbf{x}$ is a string. The length of $\mathbf{xa}$ is $\mathbf{1} + |\mathbf{x}|$

Convention

- $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ denote elements of $\boldsymbol{\Sigma}$
- $\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ denote strings
- $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ denote sets of strings

Much ado about nothing

- ϵ is a string containing no symbols. It is not a set
- $\{\epsilon\}$ is a set containing one string: the empty string. It is a set, not a string.
- $\emptyset$ is the empty set. It contains no strings.
- $\{\emptyset\}$ is a set containing one element, which itself is a set that contains no elements.

Concatenation and properties

- If $\mathbf{x}$ and $\mathbf{y}$ are strings then $\mathbf{xy}$ denotes their concatenation.
- **concatenation** defined recursively :
  - $\mathbf{xy} = \mathbf{y}$ if $\mathbf{x} = \epsilon$
  - $\mathbf{xy} = \mathbf{a}(\mathbf{wy})$ if $\mathbf{x} = \mathbf{aw}$
- $\mathbf{xy}$ sometimes written as $\mathbf{x} \bullet \mathbf{y}$.
- concatenation is associative: $(\mathbf{uv})\mathbf{w} = \mathbf{u}(\mathbf{vw})$
  hence write $\mathbf{uvw} \equiv (\mathbf{uv})\mathbf{w} = \mathbf{u}(\mathbf{vw})$
- **not** commutative: $\mathbf{uv}$ not necessarily equal to $\mathbf{vu}$
- The identity element is the empty string ϵ:

$$\epsilon \mathbf{u} = \mathbf{u}\epsilon = \mathbf{u}.$$

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Substrings, prefix, suffix

Definition 1.2.

$\mathbf{v}$ is **substring** of $\mathbf{w}$ $\iff$ there exist strings $\mathbf{x}, \mathbf{y}$ such that $\mathbf{w} = \mathbf{xvy}$.

- If $\mathbf{x} = \epsilon$ then $\mathbf{v}$ is a prefix of $\mathbf{w}$
- If $\mathbf{y} = \epsilon$ then $\mathbf{v}$ is a suffix of $\mathbf{w}$

String exponents

Definition 1.3.

If $\mathbf{w}$ is a string then $\mathbf{w^n}$ is defined inductively as follows:
$\mathbf{w^n} = \epsilon$ if $\mathbf{n = 0}$
$\mathbf{w^n} = \mathbf{ww^{n-1}}$ if $\mathbf{n > 0}$

Example: $\mathbf{(blah)^4 = blahblahblahblah}$.

Set Concatenation

Definition 1.4.

Given two sets $\mathbf{X}$ and $\mathbf{Y}$ of strings (over some common alphabet $\mathbf{\Sigma}$) the **concatenation** of $\mathbf{X}$ and $\mathbf{Y}$ is

$$\mathbf{XY} = \{\mathbf{xy} \mid \mathbf{x} \in \mathbf{X}, \mathbf{y} \in \mathbf{Y}\}$$

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Example 1.5.

$\mathbf{X} = \{\mathbf{fido}, \mathbf{rover}, \mathbf{spot}\}$,
$\mathbf{Y} = \{\mathbf{fluffy}, \mathbf{tabby}\}$
$\implies$
$\mathbf{XY} = \{\mathbf{fidofluffy}, \mathbf{fidotabby}, \mathbf{roverfluffy}, \ldots\}$.

Σ^* and languages

Definition 1.6.

1. Σ^n is the set of all strings of length n. Defined inductively:
   $\Sigma^n = \{\epsilon\}$ if $n = 0$
   $\Sigma^n = \Sigma\Sigma^{n-1}$ if $n > 0$
2. $\Sigma^* = \cup_{n\geq 0}\Sigma^n$ is the set of all finite length strings
3. $\Sigma^+ = \cup_{n\geq 1}\Sigma^n$ is the set of non-empty strings.

Definition 1.7.

A language L is a set of strings over Σ. In other words $L \subseteq \Sigma^*$.

Σ^* and languages

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A language $\mathbf{L}$ is a set of strings over $\mathbf{\Sigma}$. In other words $\mathbf{L \subseteq \Sigma^*}$.

Exercise

Answer the following questions taking $\mathbf{\Sigma = \{0, 1\}}$.

1. What is $\mathbf{\Sigma^0}$?
2. How many elements are there in $\mathbf{\Sigma^3}$?
3. How many elements are there in $\mathbf{\Sigma^n}$?
4. What is the length of the longest string in $\mathbf{\Sigma}$?
5. Does $\mathbf{\Sigma^*}$ have strings of infinite length?
6. If $|\mathbf{u}| = \mathbf{2}$ and $|\mathbf{v}| = \mathbf{3}$ then what is $|\mathbf{u} \bullet \mathbf{v}|$?
7. Let $\mathbf{u}$ be an arbitrary string in $\mathbf{\Sigma^*}$. What is $\boldsymbol{\epsilon}\mathbf{u}$? What is $\mathbf{u}\boldsymbol{\epsilon}$?
8. Is $\mathbf{uv = vu}$ for every $\mathbf{u, v} \in \mathbf{\Sigma^*}$?
9. Is $\mathbf{(uv)w = u(vw)}$ for every $\mathbf{u, v, w} \in \mathbf{\Sigma^*}$?

1.1.1
Exercise solved in detail

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1.2
Countable sets, countably infinite sets, and languages

Countable sets

Definition 1.1.

A set $\mathbf{X}$ is countable, if its elements can be counted.
There exists an injective mapping from $\mathbf{X}$ to natural numbers $\mathbf{N} = \{1, 2, 3, \ldots\}$.

Example 1.2.

All finite sets are countable: $\{\mathbf{aba}, \mathbf{ima}, \mathbf{saba}, \mathbf{safta}, \mathbf{uma}, \mathbf{upa}\}$.

Example 1.3.

$\mathbb{N} \times \mathbb{N} = \{(\mathbf{i}, \mathbf{j}) \mid \mathbf{i}, \mathbf{j} \in \mathbb{N}\}$ is countable.

: Proof: $\mathbf{f}(\mathbf{i}, \mathbf{j}) = 2^{\mathbf{i}} 3^{\mathbf{j}}$.

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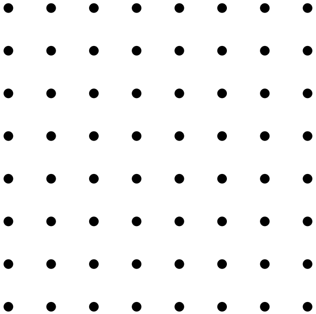
$\mathbb{N} \times \mathbb{N}$ is countable



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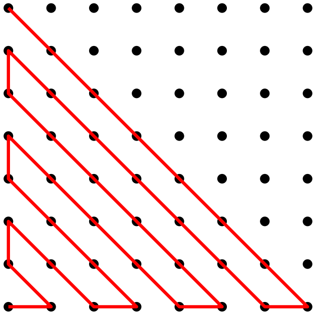

Canonical order and countability of strings

Definition 1.4.

A set $\mathbf{X}$ is countably infinite (countable and infinite) if there is a bijection $\mathbf{f}$ between the natural numbers and $\mathbf{X}$.

Alternatively: $\mathbf{X}$ is countably infinite if $\mathbf{X}$ is an infinite set and there enumeration of elements of $\mathbf{X}$.

The set of all strings is countable

Theorem 1.5.
$\boldsymbol{\Sigma}^*$ *is countable for any finite* $\boldsymbol{\Sigma}$.

Enumerate strings in order of increasing length and for each given length enumerate strings in dictionary order (based on some fixed ordering of $\boldsymbol{\Sigma}$).

Example: $\{0, 1\}^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, \ldots\}$.
$\{a, b, c\}^* = \{\epsilon, a, b, c, aa, ab, ac, ba, bb, bc, \ldots\}$

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Example: $\{\mathbf{0}, \mathbf{1}\}^* = \{\epsilon, \mathbf{0}, \mathbf{1}, \mathbf{00}, \mathbf{01}, \mathbf{10}, \mathbf{11}, \mathbf{000}, \mathbf{001}, \mathbf{010}, \ldots\}$.

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Exercise I

Question: Is $\mathbf{\Sigma^* \times \Sigma^* = \{(x, y) \mid x, y \in \Sigma^*\}}$ countable?

Question: Is $\mathbf{\Sigma^* \times \Sigma^* \times \Sigma^* = \{(x, y, z) \mid x, y, x \in \Sigma^*\}}$ countable?

Exercise I

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Exercise II

Answer the following questions taking $\Sigma = \{0, 1\}$.

1. Is a finite set countable?
2. X is countable, and the set $Y \subseteq X$, then is the set Y countable?
3. If X and Y are countable, is $X \setminus Y$ countable?
4. Are all infinite sets countably infinite?
5. If X_i is a countable infinite set, for $i = 1, \ldots, 700$, is $\cup_i X_i$ countable infinite?
6. If X_i is a countable infinite set, for $i = 1, \ldots,$, is $\cup_i X_i$ countable infinite?
7. Let X be a countable infinite set, and consider its power set

$$2^X = \{Y \mid Y \subseteq x\}.$$

The statement "the set 2^X is countable" is correct?

1.3
Inductive proofs on strings

Inductive proofs on strings

Inductive proofs on strings and related problems follow inductive definitions.

Definition 1.1.

The reverse $\mathbf{w}^R$ of a string $\mathbf{w}$ is defined as follows:

- $\mathbf{w}^R = \epsilon$ if $\mathbf{w} = \epsilon$
- $\mathbf{w}^R = \mathbf{x}^R\mathbf{a}$ if $\mathbf{w} = \mathbf{ax}$ for some $\mathbf{a} \in \mathbf{\Sigma}$ and string $\mathbf{x}$

Theorem 1.2.

Prove that for any strings $\mathbf{u}, \mathbf{v} \in \mathbf{\Sigma}^*$, $(\mathbf{uv})^R = \mathbf{v}^R\mathbf{u}^R$.

Example: $(\mathbf{dog} \bullet \mathbf{cat})^R = (\mathbf{cat})^R \bullet (\mathbf{dog})^R = \mathbf{tacgod}$.

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Example: $\mathbf{(dog \bullet cat)^R} = \mathbf{(cat)^R \bullet (dog)^R} = \mathbf{tacgod}$.

Principle of mathematical induction

Induction is a way to prove statements of the form $\forall \mathbf{n} \geq \mathbf{0}, \mathbf{P(n)}$ where $\mathbf{P(n)}$ is a statement that holds for integer $\mathbf{n}$.

Example: Prove that $\sum_{i=0}^{n} i = n(n+1)/2$ for all $\mathbf{n}$.

Induction template:

- **Base case:** Prove $\mathbf{P(0)}$
- **Induction hypothesis:** Let $\mathbf{k} > \mathbf{0}$ be an **arbitrary** integer. Assume that $\mathbf{P(n)}$ holds for any $\mathbf{n} \leq \mathbf{k}$.
- **Induction Step:** Prove that $\mathbf{P(n)}$ holds, for $\mathbf{n} = \mathbf{k} + \mathbf{1}$.

Structured induction

1. Unlike simple cases we are working with...
2. ...induction proofs also work for more complicated "structures".
3. Such as strings, tuples of strings, graphs etc.
4. See class notes on induction for details.

Proving the theorem

[redacted]

Prove that for any strings $\mathbf{u}, \mathbf{v} \in \mathbf{\Sigma}^*$, $(\mathbf{uv})^{\mathbf{R}} = \mathbf{v}^{\mathbf{R}}\mathbf{u}^{\mathbf{R}}$.

Proof: by induction.
On what?? $|\mathbf{uv}| = |\mathbf{u}| + |\mathbf{v}|$?
$|\mathbf{u}|$?
$|\mathbf{v}|$?

What does it mean “induction on $|\mathbf{u}|$”?

1.3.1: Three proofs by induction

1.3.1.1: Induction on $|\mathbf{u}|$

By induction on $|\mathbf{u}|$

[redacted]

Prove that for any strings $\mathbf{u}, \mathbf{v} \in \mathbf{\Sigma}^*$, $(\mathbf{uv})^{\mathbf{R}} = \mathbf{v}^{\mathbf{R}}\mathbf{u}^{\mathbf{R}}$.

Proof by induction on $|\mathbf{u}|$ means that we are proving the following.

Base case: Let $\mathbf{u}$ be an arbitrary string of length $\mathbf{0}$. $\mathbf{u} = \epsilon$ since there is only one such string. Then

$(\mathbf{uv})^{\mathbf{R}} = (\epsilon\mathbf{v})^{\mathbf{R}} = \mathbf{v}^{\mathbf{R}} = \mathbf{v}^{\mathbf{R}}\epsilon = \mathbf{v}^{\mathbf{R}}\epsilon^{\mathbf{R}} = \mathbf{v}^{\mathbf{R}}\mathbf{u}^{\mathbf{R}}$

Induction hypothesis: $\forall \mathbf{n} \geq \mathbf{0}$, for any string $\mathbf{u}$ of length $\mathbf{n}$:

For all strings $\mathbf{v} \in \mathbf{\Sigma}^*$, $(\mathbf{uv})^{\mathbf{R}} = \mathbf{v}^{\mathbf{R}}\mathbf{u}^{\mathbf{R}}$.

No assumption about $\mathbf{v}$, hence statement holds for all $\mathbf{v} \in \mathbf{\Sigma}^*$.

By induction on $|\mathbf{u}|$

Theorem 1.4.

Prove that for any strings $\mathbf{u}, \mathbf{v} \in \mathbf{\Sigma}^*$, $(\mathbf{uv})^{\mathbf{R}} = \mathbf{v}^{\mathbf{R}}\mathbf{u}^{\mathbf{R}}$.

Proof by induction on $|\mathbf{u}|$ means that we are proving the following.

Base case: Let $\mathbf{u}$ be an arbitrary string of length $\mathbf{0}$. $\mathbf{u} = \epsilon$ since there is only one such string. Then

$$(\mathbf{uv})^{\mathbf{R}} = (\epsilon\mathbf{v})^{\mathbf{R}} = \mathbf{v}^{\mathbf{R}} = \mathbf{v}^{\mathbf{R}}\epsilon = \mathbf{v}^{\mathbf{R}}\epsilon^{\mathbf{R}} = \mathbf{v}^{\mathbf{R}}\mathbf{u}^{\mathbf{R}}$$

Induction hypothesis: $\forall \mathbf{n} \geq \mathbf{0}$, for any string $\mathbf{u}$ of length $\mathbf{n}$:

For all strings $\mathbf{v} \in \mathbf{\Sigma}^*$, $(\mathbf{uv})^{\mathbf{R}} = \mathbf{v}^{\mathbf{R}}\mathbf{u}^{\mathbf{R}}$.

No assumption about $\mathbf{v}$, hence statement holds for all $\mathbf{v} \in \mathbf{\Sigma}^*$.

By induction on $|\mathbf{u}|$

[redacted]

Prove that for any strings $\mathbf{u}, \mathbf{v} \in \mathbf{\Sigma}^*$, $(\mathbf{uv})^R = \mathbf{v}^R\mathbf{u}^R$.

Proof by induction on $|\mathbf{u}|$ means that we are proving the following.

Base case: Let $\mathbf{u}$ be an arbitrary string of length $\mathbf{0}$. $\mathbf{u} = \epsilon$ since there is only one such string. Then

$$(\mathbf{uv})^R = (\epsilon\mathbf{v})^R = \mathbf{v}^R = \mathbf{v}^R\epsilon = \mathbf{v}^R\epsilon^R = \mathbf{v}^R\mathbf{u}^R$$

Induction hypothesis: $\forall \mathbf{n} \geq \mathbf{0}$, for any string $\mathbf{u}$ of length $\mathbf{n}$:

For all strings $\mathbf{v} \in \mathbf{\Sigma}^*$, $(\mathbf{uv})^R = \mathbf{v}^R\mathbf{u}^R$.

No assumption about $\mathbf{v}$, hence statement holds for all $\mathbf{v} \in \mathbf{\Sigma}^*$.

Inductive step

- Let $\mathbf{u}$ be an arbitrary string of length $\mathbf{n} > \mathbf{0}$. Assume inductive hypothesis holds for all strings $\mathbf{w}$ of length $< \mathbf{n}$.
- Since $|\mathbf{u}| = \mathbf{n} > \mathbf{0}$ we have $\mathbf{u} = \mathbf{ay}$ for some string $\mathbf{y}$ with $|\mathbf{y}| < \mathbf{n}$ and $\mathbf{a} \in \boldsymbol{\Sigma}$.
- Then

$$
\begin{aligned}
(uv)^R &= ((ay)v)^R \\
&= (a(yv))^R \\
&= (yv)^R a^R \\
&= (v^R y^R) a^R \\
&= v^R (y^R a^R) \\
&= v^R (ay)^R \\
&= v^R u^R
\end{aligned}
$$

Inductive step

- Let $\mathbf{u}$ be an arbitrary string of length $\mathbf{n} > \mathbf{0}$. Assume inductive hypothesis holds for all strings $\mathbf{w}$ of length $< \mathbf{n}$.
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- Then

$$
\begin{aligned}
(\mathbf{uv})^{\mathbf{R}} &= ((\mathbf{ay})\mathbf{v})^{\mathbf{R}} \\
&= (\mathbf{a}(\mathbf{yv}))^{\mathbf{R}} \\
&= (\mathbf{yv})^{\mathbf{R}}\mathbf{a}^{\mathbf{R}} \\
&= (\mathbf{v}^{\mathbf{R}}\mathbf{y}^{\mathbf{R}})\mathbf{a}^{\mathbf{R}} \\
&= \mathbf{v}^{\mathbf{R}}(\mathbf{y}^{\mathbf{R}}\mathbf{a}^{\mathbf{R}}) \\
&= \mathbf{v}^{\mathbf{R}}(\mathbf{ay})^{\mathbf{R}} \\
&= \mathbf{v}^{\mathbf{R}}\mathbf{u}^{\mathbf{R}}
\end{aligned}
$$

1.3.1.2: A failed attempt: Induction on $|\mathbf{v}|$

Induction on $|\mathbf{v}|$

[redacted]

Prove that for any strings $\mathbf{u}, \mathbf{v} \in \mathbf{\Sigma}^*$, $(\mathbf{uv})^{\mathbf{R}} = \mathbf{v}^{\mathbf{R}}\mathbf{u}^{\mathbf{R}}$.

Proof by induction on $|\mathbf{v}|$ means that we are proving the following.

Induction hypothesis: $\forall \mathbf{n} \geq \mathbf{0}$, for any string $\mathbf{v}$ of length $\mathbf{n}$:

For all strings $\mathbf{u} \in \mathbf{\Sigma}^*$, $(\mathbf{uv})^{\mathbf{R}} = \mathbf{v}^{\mathbf{R}}\mathbf{u}^{\mathbf{R}}$.

Base case: Let $\mathbf{v}$ be an arbitrary string of length $\mathbf{0}$. $\mathbf{v} = \epsilon$ since there is only one such string. Then

$$(\mathbf{uv})^{\mathbf{R}} = (\mathbf{u}\epsilon)^{\mathbf{R}} = \mathbf{u}^{\mathbf{R}} = \epsilon\mathbf{u}^{\mathbf{R}} = \epsilon^{\mathbf{R}}\mathbf{u}^{\mathbf{R}} = \mathbf{v}^{\mathbf{R}}\mathbf{u}^{\mathbf{R}}$$

Induction on $|\mathbf{v}|$

[redacted]

Prove that for any strings $\mathbf{u}, \mathbf{v} \in \mathbf{\Sigma}^*$, $(\mathbf{uv})^{\mathbf{R}} = \mathbf{v}^{\mathbf{R}}\mathbf{u}^{\mathbf{R}}$.

Proof by induction on $|\mathbf{v}|$ means that we are proving the following.
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Base case: Let $\mathbf{v}$ be an arbitrary string of length $\mathbf{0}$. $\mathbf{v} = \epsilon$ since there is only one such string. Then

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Induction on $|\mathbf{v}|$

[redacted]

Prove that for any strings $\mathbf{u}, \mathbf{v} \in \mathbf{\Sigma}^*$, $(\mathbf{uv})^\mathbf{R} = \mathbf{v}^\mathbf{R}\mathbf{u}^\mathbf{R}$.

Proof by induction on $|\mathbf{v}|$ means that we are proving the following.
Induction hypothesis: $\forall \mathbf{n} \geq \mathbf{0}$, for any string $\mathbf{v}$ of length $\mathbf{n}$:
For all strings $\mathbf{u} \in \mathbf{\Sigma}^*$, $(\mathbf{uv})^\mathbf{R} = \mathbf{v}^\mathbf{R}\mathbf{u}^\mathbf{R}$.

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Inductive step

- Let $\mathbf{v}$ be an arbitrary string of length $\mathbf{n} > \mathbf{0}$. Assume inductive hypothesis holds for all strings $\mathbf{w}$ of length $< \mathbf{n}$.
- Since $|\mathbf{v}| = \mathbf{n} > \mathbf{0}$ we have $\mathbf{v} = \mathbf{ay}$ for some string $\mathbf{y}$ with $|\mathbf{y}| < \mathbf{n}$ and $\mathbf{a} \in \mathbf{\Sigma}$.
- Then

$$\begin{aligned} (\mathbf{uv})^{\mathbf{R}} &= (\mathbf{u}(\mathbf{ay}))^{\mathbf{R}} \\ &= ((\mathbf{ua})\mathbf{y})^{\mathbf{R}} \\ &= \mathbf{y}^{\mathbf{R}}(\mathbf{ua})^{\mathbf{R}} \\ &= ?? \end{aligned}$$

Cannot simplify $(\mathbf{ua})^{\mathbf{R}}$ using inductive hypothesis. Can simplify if we extend base case to include $\mathbf{n} = \mathbf{0}$ and $\mathbf{n} = \mathbf{1}$. However, $\mathbf{n} = \mathbf{1}$ itself requires induction on $|\mathbf{u}|$!

Inductive step

- Let $\mathbf{v}$ be an arbitrary string of length $\mathbf{n} > \mathbf{0}$. Assume inductive hypothesis holds for all strings $\mathbf{w}$ of length $< \mathbf{n}$.
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1.3.1.3: Induction on $|\mathbf{u}| + |\mathbf{v}|$

Induction on $|\mathbf{u}| + |\mathbf{v}|$

[redacted]

Prove that for any strings $\mathbf{u}, \mathbf{v} \in \Sigma^*$, $(\mathbf{uv})^R = \mathbf{v}^R\mathbf{u}^R$.

Proof by induction on $|\mathbf{u}| + |\mathbf{v}|$ means that we are proving the following.

Induction hypothesis: $\forall \mathbf{n} \geq \mathbf{0}$, for any $\mathbf{u}, \mathbf{v} \in \Sigma^*$ with $|\mathbf{u}| + |\mathbf{v}| \leq \mathbf{n}$, $(\mathbf{uv})^R = \mathbf{v}^R\mathbf{u}^R$.

Base case: $\mathbf{n} = \mathbf{0}$. Let $\mathbf{u}, \mathbf{v}$ be an arbitrary strings such that $|\mathbf{u}| + |\mathbf{v}| = \mathbf{0}$. Implies $\mathbf{u}, \mathbf{v} = \epsilon$.

Inductive step: $\mathbf{n} > \mathbf{0}$. Let $\mathbf{u}, \mathbf{v}$ be arbitrary strings such that $|\mathbf{u}| + |\mathbf{v}| = \mathbf{n}$.

Induction on $|\mathbf{u}| + |\mathbf{v}|$

Theorem 1.6.

Prove that for any strings $\mathbf{u}, \mathbf{v} \in \mathbf{\Sigma}^*$, $(\mathbf{uv})^{\mathbf{R}} = \mathbf{v}^{\mathbf{R}}\mathbf{u}^{\mathbf{R}}$.

Proof by induction on $|\mathbf{u}| + |\mathbf{v}|$ means that we are proving the following.
Induction hypothesis: $\forall \mathbf{n} \geq \mathbf{0}$, for any $\mathbf{u}, \mathbf{v} \in \mathbf{\Sigma}^*$ with $|\mathbf{u}| + |\mathbf{v}| \leq \mathbf{n}$, $(\mathbf{uv})^{\mathbf{R}} = \mathbf{v}^{\mathbf{R}}\mathbf{u}^{\mathbf{R}}$.

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Induction on $|\mathbf{u}| + |\mathbf{v}|$

[redacted]

Prove that for any strings $\mathbf{u}, \mathbf{v} \in \boldsymbol{\Sigma}^*$, $(\mathbf{uv})^{\mathbf{R}} = \mathbf{v}^{\mathbf{R}}\mathbf{u}^{\mathbf{R}}$.

Proof by induction on $|\mathbf{u}| + |\mathbf{v}|$ means that we are proving the following.
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Induction on $|\mathbf{u}| + |\mathbf{v}|$

[redacted]

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1.4
Languages

Languages

[redacted]

A language $\mathbf{L}$ is a set of strings over $\mathbf{\Sigma}$. In other words $\mathbf{L} \subseteq \mathbf{\Sigma}^*$.

Standard set operations apply to languages.

- For languages $\mathbf{A}, \mathbf{B}$ the concatenation of $\mathbf{A}, \mathbf{B}$ is $\mathbf{AB} = \{\mathbf{xy} \mid \mathbf{x} \in \mathbf{A}, \mathbf{y} \in \mathbf{B}\}$.
- For languages $\mathbf{A}, \mathbf{B}$, their union is $\mathbf{A} \cup \mathbf{B}$, intersection is $\mathbf{A} \cap \mathbf{B}$, and difference is $\mathbf{A} \setminus \mathbf{B}$ (also written as $\mathbf{A} - \mathbf{B}$).
- For language $\mathbf{A} \subseteq \mathbf{\Sigma}^*$ the complement of $\mathbf{A}$ is $\bar{\mathbf{A}} = \mathbf{\Sigma}^* \setminus \mathbf{A}$.

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- For language $\mathbf{A} \subseteq \boldsymbol{\Sigma}^*$ the complement of $\mathbf{A}$ is $\bar{\mathbf{A}} = \boldsymbol{\Sigma}^* \setminus \mathbf{A}$.

Exponentiation, Kleene star etc

Definition 1.2.

For a language $\mathbf{L} \subseteq \mathbf{\Sigma}^*$ and $\mathbf{n} \in \mathbb{N}$, define $\mathbf{L}^\mathbf{n}$ inductively as follows.

$$\mathbf{L}^\mathbf{n} = \begin{cases} \{\epsilon\} & \text{if } \mathbf{n} = \mathbf{0} \\ \mathbf{L} \bullet (\mathbf{L}^{\mathbf{n}-\mathbf{1}}) & \text{if } \mathbf{n} > \mathbf{0} \end{cases}$$

And define $\mathbf{L}^* = \cup_{\mathbf{n} \geq \mathbf{0}} \mathbf{L}^\mathbf{n}$, and $\mathbf{L}^+ = \cup_{\mathbf{n} \geq \mathbf{1}} \mathbf{L}^\mathbf{n}$

Exercise

Problem 1.3.

Answer the following questions taking $\mathbf{A}, \mathbf{B} \subseteq \{0,1\}^*$.

1. *Is* $\epsilon = \{\epsilon\}$? *Is* $\emptyset = \{\epsilon\}$?
2. *What is* $\emptyset \bullet \mathbf{A}$? *What is* $\mathbf{A} \bullet \emptyset$?
3. *What is* $\{\epsilon\} \bullet \mathbf{A}$? *And* $\mathbf{A} \bullet \{\epsilon\}$?
4. *If* $|\mathbf{A}| = 2$ *and* $|\mathbf{B}| = 3$, *what is* $|\mathbf{A} \bullet \mathbf{B}|$?

Exercise

Problem 1.4.

Consider languages over $\mathbf{\Sigma = \{0, 1\}}$.

1. *What is* $\emptyset^{\mathbf{0}}$*?*
2. *If* $|\mathbf{L}| = \mathbf{2}$, *then what is* $|\mathbf{L}^4|$*?*
3. *What is* $\emptyset^*$, $\{\epsilon\}^*$, ϵ^**?*
4. *For what* $\mathbf{L}$ *is* $\mathbf{L}^*$ *finite?*
5. *What is* $\emptyset^+$, $\{\epsilon\}^+$, ϵ^+*?*

Languages and Computation

What are we interested in computing? Mostly functions.

Informal definition: An algorithm $\mathcal{A}$ computes a function $f : \Sigma^* \rightarrow \Sigma^*$ if for all $w \in \Sigma^*$ the algorithm $\mathcal{A}$ on input w terminates in a finite number of steps and outputs $f(w)$.

Examples of functions:

- Numerical functions: length, addition, multiplication, division etc
- Given graph G and s, t find shortest paths from s to t
- Given program M check if M halts on empty input
- Posts Correspondence problem

Languages and Computation

Definition 1.5.

A function $\mathbf{f}$ over $\mathbf{\Sigma^*}$ is a boolean if $\mathbf{f : \Sigma^* \to \{0, 1\}}$.

Observation: There is a bijection between boolean functions and languages.

- Given boolean function $\mathbf{f : \Sigma^* \to \{0, 1\}}$ define language $\mathbf{L_f = \{w \in \Sigma^* \mid f(w) = 1\}}$
- Given language $\mathbf{L \subseteq \Sigma^*}$ define boolean function $\mathbf{f : \Sigma^* \to \{0, 1\}}$ as follows: $\mathbf{f(w) = 1}$ if $\mathbf{w \in L}$ and $\mathbf{f(w) = 0}$ otherwise.

Languages and Computation

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- Given language $\mathbf{L} \subseteq \mathbf{\Sigma}^*$ define boolean function $\mathbf{f} : \mathbf{\Sigma}^* \rightarrow \{0, 1\}$ as follows: $\mathbf{f(w)} = 1$ if $\mathbf{w} \in \mathbf{L}$ and $\mathbf{f(w)} = 0$ otherwise.

Languages and Computation

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Language recognition problem

Definition 1.6.

For a language $\mathbf{L} \subseteq \mathbf{\Sigma}^*$ the language recognition problem associate with $\mathbf{L}$ is the following: given $\mathbf{w} \in \mathbf{\Sigma}^*$, is $\mathbf{w} \in \mathbf{L}$?

- Equivalent to the problem of "computing" the function $\mathbf{f_L}$.
- Language recognition is same as boolean function computation
- How difficult is a function $\mathbf{f}$ to compute? How difficult is the recognizing $\mathbf{L_f}$?

Why two different views? Helpful in understanding different aspects?

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How many languages are there?

The answer my friend is blowing in the slides.

Recall:

Definition 1.7.

An set $\mathbf{X}$ is countable if there is a bijection $\mathbf{f}$ between the natural numbers and $\mathbf{A}$.

Theorem 1.8.

$\mathbf{\Sigma}^*$ is countable for every finite $\mathbf{\Sigma}$.

The set of all languages is $\mathbb{P}(\mathbf{\Sigma}^*)$ the power set of $\mathbf{\Sigma}^*$

Theorem 1.9 (Cantor).

*$\mathbb{P}(\mathbf{\Sigma}^*)$ is **not** countable for any finite $\mathbf{\Sigma}$.*

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The set of all languages is $\mathbb{P}(\mathbf{\Sigma}^*)$ the power set of $\mathbf{\Sigma}^*$

[redacted]

$\mathbb{P}(\mathbf{\Sigma}^*)$ *is* **not** *countable for any finite* $\mathbf{\Sigma}$.

Cantor's diagonalization argument

[redacted]

$\mathbb{P}(\mathbb{N})$ *is not countable.*

- Suppose $\mathbb{P}(\mathbb{N})$ is countable infinite. Let $\mathbf{S_1, S_2, \ldots,}$ be an enumeration of all subsets of numbers.
- Let $\mathbf{D}$ be the following diagonal subset of numbers.

$$\mathbf{D} = \{\mathbf{i} \mid \mathbf{i} \notin \mathbf{S_i}\}$$

- Since $\mathbf{D}$ is a set of numbers, by assumption, $\mathbf{D} = \mathbf{S_j}$ for some $\mathbf{j}$.
- **Question:** Is $\mathbf{j} \in \mathbf{D}$?

Consequences for Computation

- How many **C** programs are there? The set of **C** programs is countable since each of them can be represented as a string over a finite alphabet.
- How many languages are there? Uncountably many!
- Hence some (in fact almost all!) languages/boolean functions do not have any **C** program to recognize them.

Questions:

- Maybe interesting languages/functions have **C** programs and hence computable. Only uninteresting languors uncomputable?
- Why should **C** programs be the definition of computability?
- Ok, there are difficult problems/languages. what languages are computable and which have efficient algorithms?

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- Why should **C** programs be the definition of computability?
- Ok, there are difficult problems/languages. what languages are computable and which have efficient algorithms?

Easy languages

Definition 1.11.

A language $\mathbf{L} \subseteq \mathbf{\Sigma}^*$ is finite if $|\mathbf{L}| = \mathbf{n}$ for some integer $\mathbf{n}$.

Exercise: Prove the following.

Theorem 1.12.

The set of all finite languages is countable.

1.5
Overview of whats coming on finite automata/complexity

Languages: easiest, easy, hard, really hard, really really hard

1. Finite languages.
2. Regular languages.
    2.1 Regular expressions.
    2.2 DFA: Deterministic finite automata.
    2.3 NFA: Non-deterministic finite automata.
    2.4 Languages that are not regular.
3. Context free languages (stack).
4. Turing machines: Decidable languages.
5. TM Undecidable languages (halting theorem).
6. TM Unrecognizable languages.

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