CS/ECE 374 A ✧ Fall 2021
↶ Conflict Midterm 1 ↷
September 28, 2021

↶ Directions ↷

- *Don't panic!*

- If you brought anything except your writing implements, your **hand-written** double-sided 8½" × 11" cheat sheet, please put it away for the duration of the exam. In particular, please turn off and put away *all* medically unnecessary electronic devices.

- The exam has five numbered questions.

- Write your answers on blank white paper. Please start your solution to each numbered question on a new sheet of paper.

- You have 150 minutes to write, scan, and submit your solutions. The exam is designed to take at most 120 minutes to complete. We are providing 30 minutes of slack to scan and submit in case of unforeseen technology issues.

- If you are ready to scan your solutions before 9:15pm, send a private message to the host ("Ready to scan") and wait for confirmation before leaving the Zoom call.

- Please scan *all* paper that you used during the exam — first your solutions, in the correct order, then your cheat sheet (if any), and finally any scratch paper.

- Proofs are required for full credit if and only if we explicitly ask for them, using the word *prove* in bold italics. In particular, if we ask you to show that a language is regular, you can provide a regular expression, DFA, NFA, or boolean combination *without justification*. Similarly, if we ask you to give a DFA or NFA, you to *not* have to name or describe the states.

- Finally, if something goes seriously wrong, send email to jeffe@illinois.edu as soon as possible explaining the situation. If you have already finished the exam but cannot submit to Gradescope for some reason, include a complete scan of your exam in your email. If you are in the middle of the exam, send Jeff email, finish the exam (if you can) within the time limit, and then send a second email with your completed exam.

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1. For each of the following languages over the alphabet $\Sigma = \{0, 1\}$, either prove that the language is regular (by constructing an appropriate DFA, NFA, or regular expression) or prove that the language is not regular (by constructing an infinite fooling set and proving that the set you construct is indeed a fooling set for that language).

   (a) $\{0^p 1^q 0^r \mid p = (q + r) \bmod 2\}$

   (b) $\{0^p 1^q 0^r \mid p = q + r\}$

2. Let L be any regular language over the alphabet $\Sigma = \{0, 1\}$.

   Let $\mathsf{compress}(w)$ be a function that takes a string w as input, and returns the string formed by compressing every run of 0s in w by half. Specifically, every run of $2n$ 0s is compressed to length n, and every run of $2n + 1$ 0s is compressed to length $n + 1$. For example:

   $$\mathsf{compress}(00000110001) = 00011001$$
   $$\mathsf{compress}(11000010) = 110010$$
   $$\mathsf{compress}(11111) = 11111$$

   Choose exactly one of the following languages, and prove that your chosen language is regular. (In fact, *both* languages are regular, but we only want a proof for one of them.) Don't forget to tell us which language you've chosen!

   (a) $\{w \in \Sigma^* \mid \mathsf{compress}(w) \in L\}$

   (b) $\{\mathsf{compress}(w) \mid w \in L\}$

3. Recall that the *greatest common divisor* of two positive integers p and q, written $\gcd(p, q)$, is the largest positive integer r that divides both p and q. For example, $\gcd(21, 15) = 3$ and $\gcd(3, 74) = 1$.

   Prove that the following languages are not regular by building an infinite fooling set for each of them. For each language, prove that the set you constructed is indeed a fooling set.

   (a) $\{0^p 1^q 0^r \mid p > 0 \text{ and } q > 0 \text{ and } r = \gcd(p, q)\}$.

   (b) $\{0^p 1^{pq} \mid p > 0 \text{ and } q > 0\}$

4. Consider the following recursive function, RO (short for remove-ones) that operates on any string $w \in \Sigma^*$, where $\Sigma = \{0, 1\}$:

   $$\mathsf{RO}(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ 0 \cdot \mathsf{RO}(x) & \text{if } w = 0 \cdot x \text{ for some string } x \\ \mathsf{RO}(x) & \text{if } w = 1 \cdot x \text{ for some string } x \end{cases}$$

   (a) Prove that $|\mathsf{RO}(w)| \leq |w|$ for all strings w.

   (b) Prove that $\mathsf{RO}(\mathsf{RO}(w)) = \mathsf{R0}(w)$ for all strings w.

   (There's one more question on the next page)

5. For each statement below, write "Yes" if the statement is always true and write "No" otherwise, and give a brief (one short sentence) explanation of your answer. Read these statements very carefully—small details matter!

   (a) $\{0^n 1 \mid n > 0\}$ is the only infinite fooling set for the language $\{0^n 1 0^n \mid n > 0\}$.

   (b) $\{0^n 1 0^n \mid n > 0\}$ is a context-free language.

   (c) The context-free grammar $S \to 00S \mid S11 \mid 01$ generates the language $0^n 1^n$.

   (d) Any language that can be decided by an NFA with ε-transitions can also be decided by an NFA without ε-transitions.

   (e) For any string $w \in (0+1)^*$, let w^C denote the string obtained by flipping every 0 in w to 1, and every 1 in w to 0.
   If L is a regular language over the alphabet $\{0, 1\}$, then $\{ww^C \mid w \in L\}$ is also regular.

   (f) For any string $w \in (0+1)^*$, let w^C denote the string obtained by flipping every 0 in w to 1, and every 1 in w to 0.
   If L is a regular language over the alphabet $\{0, 1\}$, then $\{xy^C \mid x, y \in L\}$ is also regular.

   (g) The ε-reach of any state in an NFA contains the state itself.

   (h) Let L_1, L_2 be two regular languages. The language $(L_1 + L_2)^*$ is also regular.

   (i) The regular expression $(00 + 11)^*$ represents the language of all strings over $\{0, 1\}$ of even length.

   (j) The language $\{0^{2p} \mid p \text{ is prime}\}$ is regular.