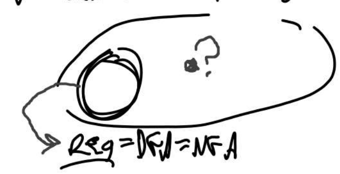
Fooling Sets

Thursday, September 10, 2020 12:59 PM

What's not regular?



Canonical example: same # of 0s followed by same # of 1's.

$L = \{ 0^n 1^n \mid n \in \mathbb{N} \}$

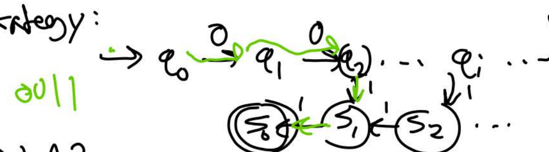
How do we show it's non regular?

$x \in L$

needs to count 100 1's.

← needs to count the # of first 0's it reads

Strategy:



0011

where I stop?

// q_i means we've read i 0's so far.

// s_i means... i 1's left to read

DFA?

PDA

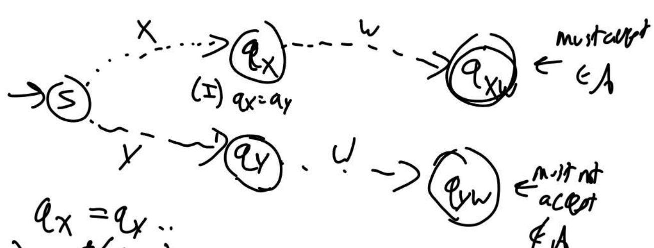
Problem: seems to require ∞ # of states.

Generalizing a bit. (Suppose $D = (\delta, Q, s, A)$ accepts L)

$x = 0^m$ $\quad y = 0^n$ $\quad m \neq n$ $\quad w = 1^m$

$q_x = \delta^*(s, x)$ State we end up in after processing first m 0s.

$q_y = \delta^*(s, y)$



(I) What if $q_x = q_y$..

$\delta^*(s,x) = \delta^*(s,y)$

$q_{xw} = \delta^*(s, xw) = \delta^*(\delta^*(s,x), w)$

$= \delta^*(\delta^*(s,y), w)$

$= \delta^*(s, yw) = q_{yw}$ $\quad$ X

(II) $q_x \neq q_y$.

This would have to hold for all choices of x, y.

Pigeonhole

Let $K = |Q|$ be the finite # of states.

Consider the $K+1$ different strings 0^i for $0 \le i \le K+1$,

There must be some pair of strings
$x = 0^m \quad y = 0^n \qquad 0 < m \le K+1 \quad 0 < n \le K+1,$
$m \ne n, \quad \& \quad \delta^*(s, x) = \delta^*(s, y)$

This contradicts Case I above.

Terminology. Let L be a language.

- Distinguishing Suffix for a pair of strings x and y $\quad x \in \Sigma^* \quad y \in \Sigma^*$
  is a string $w \in \Sigma^*$ s.t. $xw \in L$ and $yw \notin L$
  OR $xw \notin L$ and $yw \in L$
  (equiv. $xw \in L \Leftrightarrow yw \notin L$)
  (e.g. for $x = 0^m$ $y = 0^n$, $L = \{0^i 1^i\}$, $w = 1^m$ is a D.S.)
- Fooling Set: Set of strings F, such that any pair $x \in F$, $y \in F$, $x \ne y$ have a distinguishing suffix.
  (e.g. $F = \{0^i \mid 0 < i \le K+1\}$, for $L = \{0^i 1^i\}$)
- If L has an infinite size fooling set, $\Longrightarrow$ L is not regular.

Skill: Show a language is NOT regular by constructing an ∞-size Fooling Set.

Ex. $\quad L = \{ w \mid w = w^R \}$ (all palindromes)

- $F = \{0^m \mid m \in \mathbb{N}\} \longleftrightarrow |F| = \infty$
- D.S. for any pair from F.
  Let $x \in F$, $y \in F$, $x \ne y$.
  $x = 0^m \quad y = 0^n \quad m \ne n$
  $w = 10^m$

... s.t. $xw \in L$ $\quad yw \notin L$

("pumping lemma") s.t. $xw \in L$, $yw \notin L$

$xw = 0^m 1 0^m \in L$

$yw = 0^n 1 0^m \neq 0^m 1 0^n$

$yw \notin L$. ✓

→ Does F have to be $F \subseteq L$?

No.

→. For any $x, y \in F$, at least x or y has to have $xw \in L$ or $yw \in L$ for some suffix w

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Non regularity by closure properties

- From above we know $L'_r = \{0^m 1^m\}$ is non-regular.
- Now consider $L = \{w \mid \#_0(w) = \#_1(w)\}$. eg 1010

Suppose L is regular.

Ways to make more regular langs: $\cdot$ $\cup$ $\cap$ $*$

Construct L' using closure properties.

$L' = L \cap (0^* 1^*)$ ✓

If L is regular, L' is regular. Contradicts. ✓

Myhill-Nerode

Inf FS $\Rightarrow$ non regular.

No infinite FS $\Rightarrow$ regular.

Give a recipe for building a DFA based on max. size fooling sets.

Define equiv. $\equiv_L$: (general symbol for languages.)

$X \equiv_L Y$ if they have no distinguishing suffix.

b/w. $xw \in L \iff yw \in L$

Equivalence classes of $\equiv_L$ correspond to a fooling set.

$\equiv_L$ partitions Σ^*.

$X \equiv_L Z$

$X \not\equiv_L Y \Rightarrow \exists$ DS for x, y.

Any set of representatives from equiv. classes of $\equiv_L$ are a fooling set.

If $\equiv_L$ has finite # of equiv classes then we can build a DFA for L.

$$Q = \{ [\equiv_L] \}$$ equivalence classes of $\equiv_L$

$$\delta(q, a) = \underline{q'} \text{ where } \underline{x}a \in q',$$ for some $x \in q$.