Once, or twice, though you should fail,
Try, try again;
If you would, at last, prevail,
Try, try again;
If we strive, 'tis no disgrace.
Though we may not win the race;
What should you do in that case?
Try, try again.

— Thomas H. Palmer, *The Teacher's Manual: Being an Exposition of an Efficient and Economical System of Education Suited to the Wants of a Free People* (1840)

I dropped my dinner, and ran back to the laboratory. There, in my excitement, I tasted the contents of every beaker and evaporating dish on the table. Luckily for me, none contained any corrosive or poisonous liquid.

— Constantine Fahlberg on his discovery of saccharin, *Scientific American* (1886)

CHAPTER 2

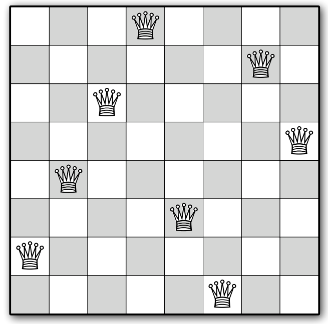
Backtracking

Still in progress. ★★★

This chapter describes another recursive algorithm strategy called ***backtracking***. A backtracking algorithm tries to build a solution to a computational problem incrementally, one small piece at a time. Whenever the algorithm needs to decide between multiple alternatives to the next component of the solution, it simply tries all possible options recursively.

2.1 *n* Queens

The prototypical backtracking problem is the classical ***n* Queens Problem**, first proposed by German chess enthusiast Max Bezzel in 1848 (under his pseudonym "Schachfreund") for the standard 8×8 board and by François-Joseph Eustache Lionnet in 1869 for the more general $n \times n$ board. The problem is to place n queens on an $n \times n$ chessboard, so that no two queens can attack each other. For readers not familiar with the rules of chess, this means that no two queens are in the same row, column, or diagonal.



One solution to the 8 queens problem, represented by the array $[4,7,3,8,2,5,1,6]$

In a letter written to his friend Heinrich Schumacher in 1850, Gauss wrote that one could easily confirm Franz Nauck's claim that the Eight Queens problem has 92 solutions by trial and error in a few hours. ("*Schwer ist es übrigens nicht, durch ein methodisches Tatonnieren sich diese Gewifsheit zu verschaffen, wenn man eine oder ein paar Stunden daran wenden will.*") His description *Tatonnieren* comes from the French *tâttoner*, meaning to feel, grope, or fumble around blindly, as if in the dark. Unfortunately, Gauss did not describe the mechanical groping method he had in mind, but he did observe that any solution can be represented by a permutation of the integers 1 through 8 satisfying a few simple arithmetic properties.

Following Gauss, let's represent possible solutions to the n-queens problem using an array $Q[1..n]$, where $Q[i]$ indicates which square in row i contains a queen. Then we can find solutions using the following recursive strategy, described in 1882 by the French recreational mathematician Édouard Lucas, who attributed the method to Emmanuel Laquière.[1] We place queens on the board one row at a time, starting at the top. To place the rth queen, we try all n squares in row r from left to right in a simple for loop. If a particular square is attacked by an earlier queen, we ignore that square; otherwise, we tentatively place a queen on that square and *recursively* grope for consistent placements of the queens in later rows.

Figure 2.1 shows the resulting algorithm, which recursively enumerates *all* complete n-queens solutions that are consistent with a given partial solution. The input parameter r is the first empty row; thus, to compute all n-queens solutions with no restrictions, we would call RECURSIVENQUEENS$(Q[1..n], 1)$. The outer for-loop considers all possible placements of a queen on row r; the inner for-loop checks whether a candidate placement of row r is consistent with the queens that are already on the first $r-1$ rows.

The execution of RECURSIVENQUEENS can be illustrated using a ***recursion tree***. Each node in this tree corresponds to a legal partial solution; in particular, the root corresponds to the empty board (with $r=0$). Edges in the recursion tree correspond to recursive calls. Leaves correspond to partial solutions that cannot be further extended,

[1]Édouard Lucas. Quatrième recreation: Le probléme des huit-reines au jeu des èchecs. Chapter 4 in *Récréations Mathématiques*, 1882.

```
RecursiveNQueens(Q[1..n], r):
  if r = n + 1
      print Q
  else
      for j ← 1 to n
          legal ← True
          for i ← 1 to r − 1
              if (Q[i] = j) or (Q[i] = j + r − i) or (Q[i] = j − r + i)
                  legal ← False
          if legal
              Q[r] ← j
              RecursiveNQueens(Q[1..n], r + 1)
```

Figure 2.1. Laquière's backtracking algorithm for the n-queens problem.

either because there is already a queen on every row, or because every position in the next empty row is attacked by an existing queen. The backtracking search for complete solutions is equivalent to a depth-first search of this tree.

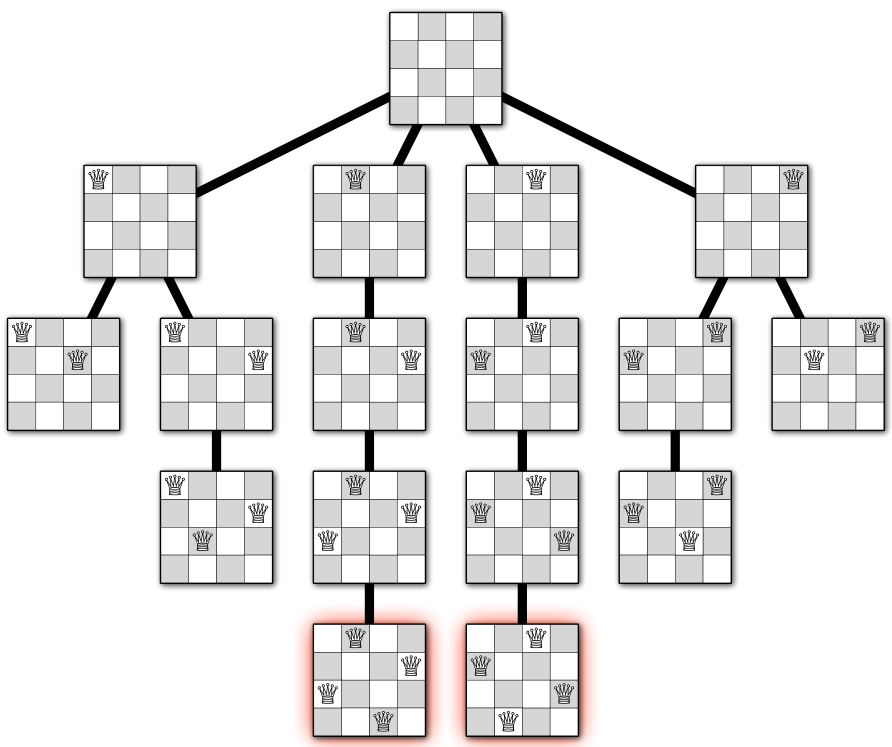


Figure 2.2. The complete recursion tree for Laquière's algorithm for the 4 queens problem.

2.2 Game Trees

Consider the following simple two-player game[2] played on an $n \times n$ square grid with a border of squares; let's call the players Horatio Fahlberg-Remsen and Vera Rebaudi.[3] Each player has n tokens that they move across the board from one side to the other. Horatio's tokens start in the left border, one in each row, and move *hor*izontally to the right; symmetrically, Vera's tokens start in the top border, one in each column, and move *ver*tically downward. The players alternate turns. In each of his turns, Horatio either *moves* one of his tokens one step to the right into an empty square, or *jumps* one of his tokens over exactly one of Vera's tokens into an empty square two steps to the right. If no legal moves or jumps are available, Horatio simply passes. Similarly, Vera either moves or jumps one of her tokens downward in each of her turns, unless no moves or jumps are possible. The first player to move all their tokens off the edge of the board wins. (It's not hard to prove that as long as there are tokens on the board, at least one player has a legal move.)

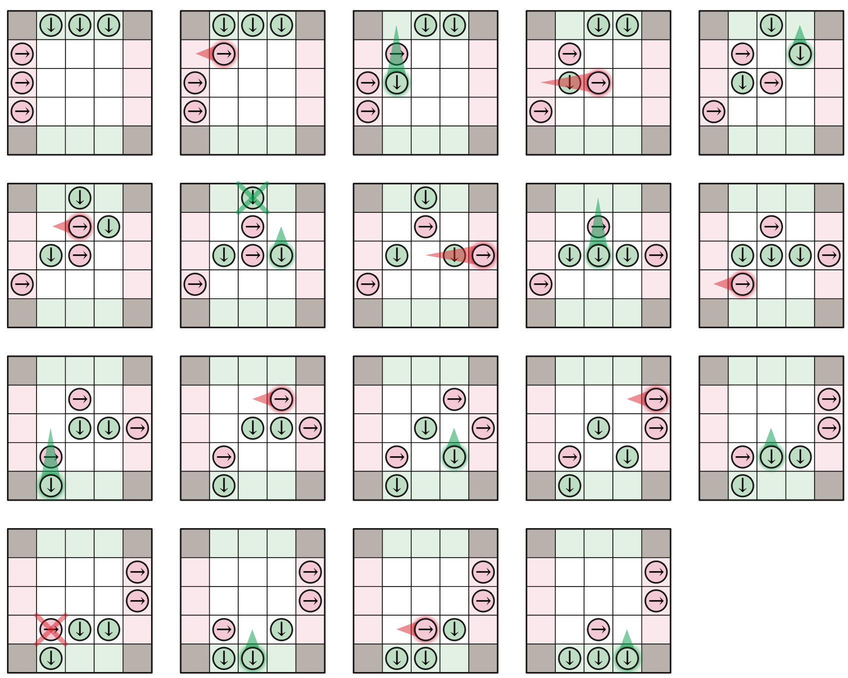


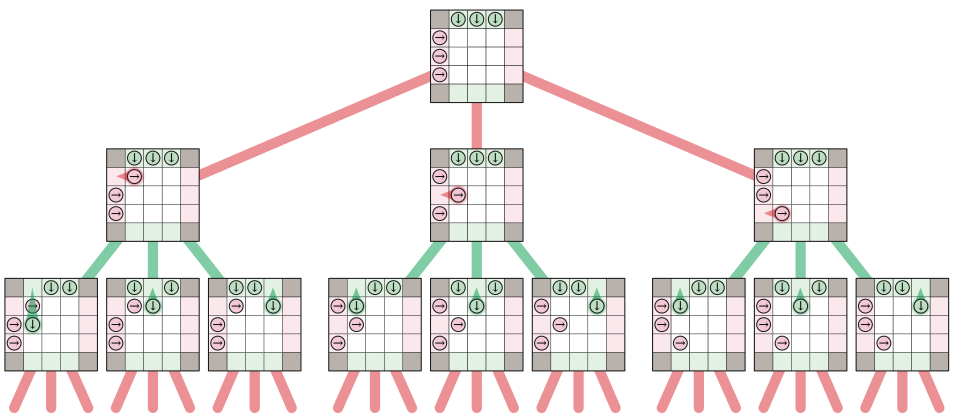
Figure 2.3. Vera wins the 3×3 fake-sugar-packet game.

[2]I don't know what this game is called, or even if I'm remembering the rules correctly; I learned it (or something like it) from Lenny Pitt, who recommended playing it with fake-sugar packets at restaurants.

[3]Constantin Fahlberg and Ira Remsen synthesized saccharin for the first time in 1878, while Fahlberg was a postdoc in Remsen's lab investigating coal tar derivatives. In 1900, Ovidio Rebaudi published the first chemical analysis of *ka'a he'ê*, a medicinal plant cultivated by the Guaraní for more than 1500 years, now more commonly known as *Stevia rebaudiana*.

Unless you've seen this game before[4], you probably don't have any idea how to play it well. Nevertheless, there is a relatively simple backtracking algorithm that can play this game—or any two-player game without randomness or hidden information—*perfectly*. That is, if we drop you into the middle of a game, and it is *possible* to win against another perfect player, the algorithm will tell you how to win.

A ***state*** of the game consists of the locations of all the pieces and the identity of the current player. These states can be connected into a *game tree*, which has an edge from state x to state y if and only if the current player in state x can legally move to state y. The root of the game tree is the initial position of the game, and every path from the root to a leaf is a complete game.



The first two levels of the fake-sugar-packet game tree.

In order to navigate through this game tree, we recursively define a game state to be ***good*** or ***bad*** as follows:

- A game state is *good* if either the current player has already won, or if the current player can move to a bad state for the opposing player.
- A game state is *bad* if either the current player has already lost, or if every available move leads to a good state for the opposing player.

Equivalently, a non-leaf node in the game tree is good if it has at least one bad child, and a non-leaf node is bad if all its children are good. By induction, any player that finds the game in a good state on their turn can win the game, even if their opponent plays perfectly; on the other hand, starting from a bad state, a player can win only if their opponent makes a mistake.

This recursive definition immediately suggests a recursive backtracking algorithm, shown Figure 2.4, to determine whether a given game state is good or bad. At its core, this algorithm is just a depth-first search of the game tree. A simple modification of this algorithm finds a good move (or even all possible good moves) if the input is a good game state.

[4]If you have, please tell me where!

```
PLAYANYGAME(X, player):
  if player has already won in state X
      return GOOD
  if player has already lost in state X
      return BAD
  for all legal moves X ⇝ Y
      if PLAYANYGAME(Y, ¬player) = BAD
          return GOOD
  return BAD
```

Figure 2.4. How to play any game perfectly.

All game-playing programs are ultimately based on this simple backtracking strategy. However, since most games have an enormous number of states, it is not possible to traverse the entire game tree in practice. Instead, game programs employ other heuristics[5] to *prune* the game tree, by ignoring states that are obviously (or "obviously") good or bad, or at least better or worse than other states, and/or by cutting off the tree at a certain depth (or *ply*) and using a more efficient heuristic to evaluate the leaves.

2.3 Subset Sum

Let's consider a more complicated problem, called SUBSETSUM: Given a set X of positive integers and *target* integer T, is there a subset of elements in X that add up to T? Notice that there can be more than one such subset. For example, if $X = \{8, 6, 7, 5, 3, 10, 9\}$ and $T = 15$, the answer is TRUE, thanks to the subsets $\{8, 7\}$ and $\{7, 5, 3\}$ and $\{6, 9\}$ and $\{5, 10\}$. On the other hand, if $X = \{11, 6, 5, 1, 7, 13, 12\}$ and $T = 15$, the answer is FALSE.

There are two trivial cases. If the target value T is zero, then we can immediately return TRUE, because empty set is a subset of *every* set X, and the elements of the empty set add up to zero.[6] On the other hand, if $T < 0$, or if $T \neq 0$ but the set X is empty, then we can immediately return FALSE.

For the general case, consider an arbitrary element $x \in X$. (We've already handled the case where X is empty.) There is a subset of X that sums to T if and only if one of the following statements is true:

- There is a subset of X that *includes* x and whose sum is T.
- There is a subset of X that *excludes* x and whose sum is T.

In the first case, there must be a subset of $X \setminus \{x\}$ that sums to $T - x$; in the second case, there must be a subset of $X \setminus \{x\}$ that sums to T. So we can solve SUBSETSUM(X, T) by reducing it to two simpler instances: SUBSETSUM$(X \setminus \{x\}, T - x)$ and SUBSETSUM$(X \setminus \{x\}, T)$. The resulting recursive algorithm is shown in Figure 2.5.

[5]A heuristic is an algorithm that doesn't work, except in practice, sometimes.

[6]... because what else could they add up to?

```
⟨⟨Does any subset of X sum to T?⟩⟩
SubsetSum(X, T):
  if T = 0
      return True
  else if T < 0 or X = ∅
      return False
  else
      x ← any element of X
      return (SubsetSum(X \ {x}, T) ∨ SubsetSum(X \ {x}, T − x))
```

Figure 2.5. A recursive backtracking algorithm for SubsetSum.

Correctness

Proving this algorithm correct is a straightforward exercise in induction. If $T = 0$, then the elements of the empty subset sum to T, so True is the correct output. Otherwise, if T is negative or the set X is empty, then no subset of X sums to T, so False is the correct output. Otherwise, if there is a subset that sums to T, then either it contains $X[n]$ or it doesn't, and the Recursion Fairy correctly checks for each of those possibilities. Done.

Analysis

In order to analyze the algorithm, we have to be a bit more precise about a few implementation details. To begin, let's assume that the input sequence X is given as an array $X[1\,..\,n]$.

The algorithm in Figure 2.5 allows us to choose *any* element $x \in X$ in the main recursive case. Purely for the sake of efficiency, it is helpful to choose an element x such that the remaining subset $X \setminus \{x\}$ has a concise representation, which can be computed quickly, so that we pay minimal overhead making the recursive calls. Specifically, we will let x be the last element $X[n]$; then the subset $X \setminus \{x\}$ is stored in the contiguous subarray $X[1\,..\,n-1]$. Passing a complete *copy* of this subarray to the recursive calls would take too long—we need $\Theta(n)$ time just to make the copy—so instead, we push only two values: the starting address of the subarray and its length. The resulting algorithm is shown in Figure 2.6. Alternatively, we could avoid passing the same starting address X to *every* recursive call by making X a global variable.

With these implementation choices, the running time $T(n)$ of our algorithm satisfies the recurrence $T(n) \le 2T(n-1) + O(1)$. From its resemblance to the Tower of Hanoi recurrence, we can guess the solution $T(n) = O(2^n)$; verifying this solution is another easy induction exercise. (We can also derive the solution directly, using either recursion trees or annihilators, as described in the appendix.) In the worst case—for example, when T is larger than the sum of all elements of X—the recursion tree for this algorithm is a complete binary tree with depth n, and the algorithm considers all 2^n subsets of X.

```
⟨⟨Does any subset of X[1..i] sum to T?⟩⟩
SubsetSum(X, i, T):
  if T = 0
      return True
  else if T < 0 or i = 0
      return False
  else
      return (SubsetSum(X, i − 1, T) ∨ SubsetSum(X, i − 1, T − X[i]))
```

Figure 2.6. A more concrete recursive backtracking algorithm for SUBSETSUM.

Variants

With only minor changes, we can solve several variants of SUBSETSUM. For example, Figure 2.7 shows an algorithm that actually *constructs* a subset of X that sums to T, if one exists, or returns the error value NONE if no such subset exists; this algorithm uses exactly the same recursive strategy as the decision algorithm in Figure 2.5. This algorithm also runs in $O(2^n)$ time; the analysis is simplest if we assume a set data structure that allows us to insert a single element in $O(1)$ time (for example, a singly-linked list), but in fact the running time is still $O(n)$ even if adding an element to Y in the second-to-last time requires $O(|Y|)$ time. Similar variants allow us to count subsets that sum to a particular value, or choose the *best* subset (according to some other criterion) that sums to a particular value.

```
⟨⟨Return a subset of X[1..i] that sums to T⟩⟩
⟨⟨or None if no such subset exists⟩⟩
ConstructSubset(X, i, T):
  if T = 0
      return ∅
  if T < 0 or n = 0
      return None
  Y ← ConstructSubset(X, i − 1, T)
  if Y ≠ None
      return Y
  Y ← ConstructSubset(X, i − 1, T − X[i])
  if Y ≠ None
      return Y ∪ {X[i]}
  return None
```

Figure 2.7. A recursive backtracking algorithm for the construction version of SUBSETSUM.

Most other problems that are solved by backtracking have this property: the same recursive strategy can be used to solve many different variants of the same problem. For example, it is easy to modify the recursive strategy described in the previous section to determine whether a given game position is good or bad to compute a move, or even the best possible move. For this reason, when we design backtracking algorithms, we should aim for the simplest possible variant of the problem, computing a number or even a single bit instead of more complex information or structure.

2.4 The General Pattern

Backtracking algorithms are commonly used to make a *sequence of decisions*, with the goal of building a recursively defined structure satisfying certain constraints; often this goal structure is itself a sequence. For example:

- In the n-queens problem, the goal is a sequence of queen positions, one in each row, such that no two queens attack each other. For each row, the algorithm *decides* where to place the queen.
- In the game tree problem, the goal is a sequence of legal moves, such that each move is as good as possible for the player making it. For each game state, the algorithm *decides* the best possible next move.
- In the subset sum problem, the goal is a sequence of input elements that have a particular sum. For each input element, the algorithm *decides* whether to include it in the output sequence or not.

(Hang on, why is the goal of *subset* sum finding a *sequence*? That was a deliberate design decision. We *impose* a convenient ordering on the input set—by representing it using an array as opposed to some other more amorphous data structure—that we can *exploit* in our recursive algorithm.)

In each recursive call to the backtracking algorithm, we need to make ***exactly one*** decision, and our choice must be consistent with all previous decisions. Thus, each recursive call requires not only the portion of the input data we have not yet processed, but also a suitable summary of the decisions we have already made. For the same of efficiency, the summary of past decisions should be as small as possible. For example:

- For the n-queens problem, we must pass in not only the number of empty rows, but the positions of all previously placed queens. We have no choice but to remember our past decisions in complete detail.
- For the game tree problem, we only need to pass in the current state of the game, including the identity of the next player. We don't need to remember anything about our past decisions, because who wins from a given game state does not depend on the moves that created that state.[7]
- For the subset sum problem, we need to pass in both the remaining available integers and the remaining target value, which is the original target value minus the *sum* of

[7] This requirement is not satisfied by all games. For example, the standard rules of chess allow a player to declare a draw if the same arrangement of pieces occurs three times, and the Chinese rules for go simply forbid repeating any earlier arrangement of stones. Thus, for these games, a game state formally includes the entire history of previous moves.

the previously chosen elements. Precisely which elements were previously chosen is unimportant.

When we design new recursive backtracking algorithms, we must figure out *in advance* what information we will need about past decisions *in the middle of the algorithm*. If this information is nontrivial, our recursive algorithm must solve a more general problem than the one we were originally asked to solve. (We've seen this kind of generalization before: To find the *median* of an unsorted array in linear time, we derived an algorithm to find the kth smallest element for *arbitrary* k.)

2.5 Longest Increasing Subsequence

For any sequence S, a ***subsequence*** of S is another sequence from S obtained by deleting zero or more elements, without changing the order of the remaining elements; the elements of the subsequence need not be together in the original sequence S. For example, when you drive down a major street in any city, you drive through a *sequence* of intersections with traffic lights, but you only have to stop at a *subsequence* of those intersections, where the traffic lights are red. If you're very lucky, you never stop at all: the empty sequence is a subsequence of S. On the other hand, if you're very unlucky, you may have to stop at every intersection: S is a subsequence of itself.

As another example, the strings `BENT`, `ACKACK`, `SQUARING`, and `SUBSEQUENT` are all subsequences of the string `SUBSEQUENCEBACKTRACKING`, as are the empty string and the entire string `SUBSEQUENCEBACKTRACKING`, but the strings `QUEUE` and `DIMAGGIO` are not. A subsequence whose elements are contiguous in the original sequence is called a ***substring***; for example, `MASHER` and `LAUGHTER` are both subsequences of `MANSLAUGHTER`, but only `LAUGHTER` is a substring.

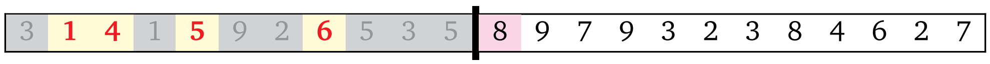
Now suppose we are given a sequence of *integers*, and we need to find the longest subsequence whose elements are in increasing order. More concretely, the input is an integer array $A[1..n]$, and we need to compute the longest possible sequence of indices $1 \le i_1 < i_2 < \cdots < i_\ell \le n$ such that $A[i_k] < A[i_{k+1}]$ for all k.

A natural approach to building this ***longest increasing subsequence*** is to *decide*, for each index j in order from 1 to n, whether or not to include $A[j]$ in the subsequence. Jumping into the middle of this decision sequence, we might imagine the following picture:

3	**1**	**4**	1	**5**	9	2	**6**	5	3	5	8	9	7	9	3	2	3	8	4	6	2	7

The black bar separates our past decisions from the portion of the input we have not yet processed. Numbers we have already decided to include are highlighted; numbers we have decided to exclude are grayed out. (Notice that the numbers we've decided to include are increasing!) Our algorithm must decide whether or not to include the number immediately after the black bar.

In this example, we *cannot* include 5, because then the selected numbers would no longer be in increasing order. So let's skip ahead to the next decision:

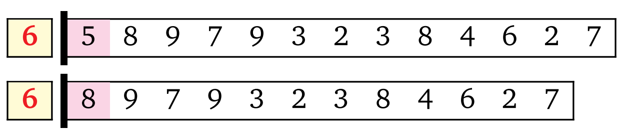


Now we *can* include 8, but it's not obvious whether we *should*. Rather than trying to be "smart", our backtracking algorithm will use simple brute force.

- First *tentatively* include the 8, and let the Recursion Fairy make the rest of the decisions.
- Then *tentatively* exclude the 8, and let the Recursion Fairy make the rest of the decisions.

Whichever choice leads to a longer increasing subsequence is the right one. (This is preciely the same recursion pattern we used to solve the subset sum problem.)

Now for the key question: *What do we need to remember about our past decisions?* We can only include $A[j]$ if the resulting subsequence is in increasing order. If we assume (inductively!) that the numbers previously selected from $A[1\,..\,j-1]$ are in increasing order, then we can include $A[j]$ if and only if $A[j]$ is larger than the last number selected from $A[1\,..\,j-1]$. Thus, the only information we need about the past is ***the last number selected so far***. We can now revise our pictures by erasing everything we don't need:



So the problem our recursive strategy is *actually* solving is the following:

> Given an integer *prev* and an array $A[1\,..\,n]$, find the longest increasing subsequence of A in which every element is larger than *prev*.

To get a complete recursive algorithm, we need a base case. Our recursive strategy breaks down when we get to the end of the array, because there is no next number to consider. But if the remaining array is empty, then the only subsequence of that array is empty, so the longest increasing subsequence of the remaining array has length 0.

Putting all the pieces together, our recursive strategy gives us the following algorithm:

```
LISBigger(prev, A[1..n]):
  if n = 0
      return 0
  else if A[1] ≤ prev
      return LISBigger(prev, A[2..n])
  else
      x ← LISBigger(prev, A[2..n])
      y ← 1 + LISBigger(A[1], A[2..n])
      return max{x, y}
```

Finally, we need to connect our recursive strategy to the original problem: Finding the longest increasing subsequence of an array *with no other constraints*. The simplest approach is to use an artificial sentinel value $-\infty$:

```
LIS(A[1..n]):
  return LISBIGGER(−∞, A[1..n])
```

Assuming we can pass around the arrays in constant time, the running time of this algorithm satisfies the recurrence $T(n) \le 2T(n-1) + O(1)$, which as usual implies that $T(n) = O(2^n)$. We really shouldn't be surprised by this running time; in the worst case, the algorithm examines each of the 2^n subsequences of the input array.

Index Formulation

In practice, passing arrays as input parameters to algorithm is rather slow; we should really find a more compact way to describe our recursive subproblems. *for purposes of designing the algorithm*, it's often useful to treat the original input array $A[1..n]$ as a global variable[8] and then reformulate the problem we're trying to solve in terms of array indices instead of explicit subarrays.

For our longest increasing subsequence problem, the integer *prev* is typically an array element $A[i]$, and the remaining array is always a suffix $A[j..n]$ of the original input array. So we can reformulate our recursive problem as follows:

> Given two indices i and j, where $i < j$, find the longest increasing subsequence of $A[j..n]$ in which every element is larger than $A[i]$.

Let $LIS(i,j)$ denote the length of the longest increasing subsequence of $A[j..n]$ with all elements larger than $A[i]$. Our recursive strategy gives us the following recurrence:

$$LIS(i,j) = \begin{cases} 0 & \text{if } j > n \\ LIS(i, j+1) & \text{if } A[i] \ge A[j] \\ \max\{LIS(i, j+1), 1 + LIS(j, j+1)\} & \text{otherwise} \end{cases}$$

To solve the original problem, we can add a sentinel value $A[0] = -\infty$ to the input array, and then recursively compute $LIS(0,1)$ following the recurrence above.

This is precisely the same algorithm as LISBIGGER; the only thing we've changed is notation. However, using index notation instead of array notation will be important when we start discussing dynamic programming in the next chapter.

★★★

> **Alternative strategy:** Let $LIS(i)$ denote the length of the longest increasing subsequence of $A[i..n]$ that begins with $A[i]$. If we set $A[0] = -\infty$, then we want $LIS(0) - 1$. In fact, this is how longest increasing subsequences are computed in $O(n \log n)$ time.

[8] In practice, we would more likely pass a pointer/reference to the original input array as another parameter of our recursive subroutine.

2.6 Optimal Binary Search Trees

Our final example combines recursive backtracking with the divide-and-conquer strategy. Recall that the running time for a successful search in a binary search tree is proportional to the number of ancestors of the target node.[9] As a result, the worst-case search time is proportional to the depth of the tree. Thus, to minimize the worst-case search time, the height of the tree should be as small as possible; by this metric, the ideal tree is perfectly balanced.

In many applications of binary search trees, however, it is more important to minimize the total cost of several searches rather than the worst-case cost of a single search. If x is a more frequent search target than y, we can save time by building a tree where the depth of x is smaller than the depth of y, even if that means increasing the overall depth of the tree. A perfectly balanced tree is *not* the best choice if some items are significantly more popular than others. In fact, a totally unbalanced tree with depth $\Omega(n)$ might actually be the best choice!

This situation suggests the following problem. Suppose we are given a sorted array of ***keys*** $A[1..n]$ and an array of corresponding ***access frequencies*** $f[1..n]$. Our task is to build the binary search tree that minimizes the *total* search time, assuming that there will be exactly $f[i]$ searches for each key $A[i]$.

Before we think about how to solve this problem, we should first come up with a good recursive definition of the function we are trying to optimize! Suppose we are also given a binary search tree T with n nodes. Let v_i denote the node that stores $A[i]$, and let r be the index of the root node. Then ignoring constant factors, the total cost of performing all the binary searches is given by the following expression:

$$Cost(T, f[1..n]) = \sum_{i=1}^{n} f[i] \cdot \#\text{ancestors of } v_i \text{ in } T \tag{$*$}$$

The root v_r is an ancestor of every node in the tree. If $i < r$, then all ancestors of v_i in the left subtree; similarly, if $i > r$, all other ancestors of v_i are in the right subtree. Thus, we can partition the cost function into three parts as follows:

$$\begin{aligned} Cost(T, f[1..n]) = \sum_{i=1}^{n} f[i] \; &+ \sum_{i=1}^{r-1} f[i] \cdot \#\text{ancestors of } v_i \text{ in } \mathit{left}(T) \\ &+ \sum_{i=r+1}^{n} f[i] \cdot \#\text{ancestors of } v_i \text{ in } \mathit{right}(T) \end{aligned}$$

[9] An *ancestor* of a node v is either the node itself or an ancestor of the parent of v. A *proper* ancestor of v is either the parent of v or a proper ancestor of the parent of v.

Now the second and third summations look exactly like our original definition $(*)$ for $Cost(T, f[1..n])$. Simple substitution now gives us a recurrence for $Cost$:

$$Cost(T, f[1..n]) = \sum_{i=1}^{n} f[i] \;+\; Cost(left(T), f[1..r-1]) \\ \;+\; Cost(right(T), f[r+1..n])$$

The base case for this recurrence is, as usual, $n = 0$; the cost of performing no searches in the empty tree is zero.

Now our task is to compute the tree T_{opt} that minimizes this cost function. Suppose we somehow magically knew that the root of T_{opt} is v_r. Then the recursive definition of $Cost(T, f)$ immediately implies that the left subtree $left(T_{\text{opt}})$ must be the optimal search tree for the keys $A[1..r-1]$ and access frequencies $f[1..r-1]$. Similarly, the right subtree $right(T_{\text{opt}})$ must be the optimal search tree for the keys $A[r+1..n]$ and access frequencies $f[r+1..n]$. **Once we choose the correct key to store at the root, the Recursion Fairy automatically constructs the rest of the optimal tree.**

More generally, let $OptCost(i, k)$ denote the total cost of the optimal search tree for the frequencies $f[i..k]$. This function obeys the following recurrence.

$$OptCost(i,k) = \begin{cases} 0 & \text{if } i > k \\ \displaystyle\sum_{j=i}^{k} f[i] + \min_{i \le r \le k} \Big\{ OptCost(i, r-1) \;+\; OptCost(r+1, k) \Big\} & \text{otherwise} \end{cases}$$

The base case correctly indicates that the minimum possible cost to perform zero searches into the empty set is zero!

This recursive definition can be translated mechanically into a recursive backtracking algorithm, whose running time is, not surprisingly, exponential. In the next chapter, we'll see how to reduce the running time to polynomial.

*Detailed time analysis

The running time of the previous backtracking algorithm obeys the recurrence

$$T(n) = \Theta(n) + \sum_{k=1}^{n} \big(T(k-1) + T(n-k) \big).$$

The $\Theta(n)$ term comes from computing the total number of searches $\sum_{i=1}^{n} f[i]$. Yeah, that's one ugly recurrence, but it's easier to solve than it looks. To transform it into a more familiar form, we regroup and collect identical terms, subtract the recurrence for

$T(n-1)$ to get rid of the summation, and then regroup again.

$$\begin{aligned} T(n) &= \Theta(n) + 2\sum_{k=0}^{n-1} T(k) \\ T(n-1) &= \Theta(n-1) + 2\sum_{k=0}^{n-2} T(k) \\ T(n) - T(n-1) &= \Theta(1) + 2T(n-1) \\ T(n) &= 3T(n-1) + \Theta(1) \end{aligned}$$

The solution $\boldsymbol{T(n) = \Theta(3^n)}$ now follows immediately by induction.

Let me emphasize that our recursive algorithm does *not* examine all possible binary search trees! The number of binary search trees with n nodes satisfies the recurrence

$$N(n) = \sum_{r=1}^{n-1} \big(N(r-1) \cdot N(n-r)\big),$$

which has the closed-from solution $N(n) = \Theta(4^n/\sqrt{n})$. Our algorithm saves considerable time by searching *independently* for the optimal left and right subtrees. A full enumeration of binary search trees would consider all possible *pairings* of left and right subtrees; hence the product in the recurrence for $N(n)$.

Exercises

1. Describe and analyze algorithms for the following generalizations of SUBSETSUM:

   (a) Given an array $X[1..n]$ of positive integers and an integer T, compute the *number* of subsets of X whose elements sum to T.

   (b) Given two arrays $X[1..n]$ and $W[1..n]$ of positive integers and an integer T, where each $W[i]$ denotes the *weight* of the corresponding element $X[i]$, compute the *maximum weight* subset of X whose elements sum to T. sIf no subset of X sums to T, your algorithm should return $-\infty$.

2. (a) Let $A[1..m]$ and $B[1..n]$ be two arbitrary arrays. A *common subsequence* of A and B is both a subsequence of A and a subsequence of B. Give a simple recursive definition for the function *lcs*(A, B), which gives the length of the *longest* common subsequence of A and B.

   (b) Let $A[1..m]$ and $B[1..n]$ be two arbitrary arrays. A *common supersequence* of A and B is another sequence that contains both A and B as subsequences. Give a simple recursive definition for the function *scs*(A, B), which gives the length of the *shortest* common supersequence of A and B.

(c) Call a sequence $X[1..n]$ *oscillating* if $X[i] < X[i+1]$ for all even i, and $X[i] > X[i+1]$ for all odd i. Give a simple recursive definition for the function *los*(A), which gives the length of the longest oscillating subsequence of an arbitrary array A of integers.

(d) Give a simple recursive definition for the function *sos*(A), which gives the length of the shortest oscillating supersequence of an arbitrary array A of integers.

(e) Call a sequence $X[1..n]$ *accelerating* if $2 \cdot X[i] < X[i-1] + X[i+1]$ for all i. Give a simple recursive definition for the function *lxs*(A), which gives the length of the longest accelerating subsequence of an arbitrary array A of integers.

For more backtracking exercises, see the next chapter!