23.2
Reducing **3-SAT** to Independent Set

Independent Set

Problem: Independent Set

Instance: A graph G, integer k.
Question: Is there an independent set in G of size k?

Lemma 23.1.

Independent set is in **NP***.*

Independent Set

Problem: Independent Set

Instance: A graph G, integer k.
Question: Is there an independent set in G of size k?

Lemma 23.1.

Independent set *is in* **NP**.

3SAT $\leq_P$ Independent Set

The reduction 3SAT $\leq_P$ Independent Set

Input: Given a $3\mathrm{CNF}$ formula φ

Goal: Construct a graph G_φ and number k such that G_φ has an independent set of size k if and only if φ is satisfiable.

G_φ should be constructable in time polynomial in size of φ

Importance of reduction: Although 3SAT is much more expressive, it can be reduced to a seemingly specialized Independent Set problem.

Notice: We handle only $3\mathrm{CNF}$ formulas – reduction would not work for other kinds of boolean formulas.

3SAT $\leq_P$ Independent Set

The reduction 3SAT $\leq_P$ Independent Set

Input: Given a $3\mathrm{CNF}$ formula φ
Goal: Construct a graph G_φ and number k such that G_φ has an independent set of size k if and only if φ is satisfiable.
G_φ should be constructable in time polynomial in size of φ

Importance of reduction: Although **3SAT** is much more expressive, it can be reduced to a seemingly specialized Independent Set problem.

Notice: We handle only $3\mathrm{CNF}$ formulas – reduction would not work for other kinds of boolean formulas.

3SAT $\leq_P$ Independent Set

The reduction 3SAT $\leq_P$ Independent Set

Input: Given a $3\mathrm{CNF}$ formula φ
Goal: Construct a graph G_φ and number k such that G_φ has an independent set of size k if and only if φ is satisfiable.
G_φ should be constructable in time polynomial in size of φ

Importance of reduction: Although **3SAT** is much more expressive, it can be reduced to a seemingly specialized Independent Set problem.

Notice: We handle only $3\mathrm{CNF}$ formulas – reduction would not work for other kinds of boolean formulas.

Interpreting **3SAT**

There are two ways to think about **3SAT**

1. Find a way to assign 0/1 (false/true) to the variables such that the formula evaluates to true, that is each clause evaluates to true.
2. Pick a literal from each clause and find a truth assignment to make all of them true. You will fail if two of the literals you pick are in conflict, i.e., you pick x_i and $\neg x_i$

We will take the second view of **3SAT** to construct the reduction.

Interpreting 3SAT

There are two ways to think about **3SAT**

1. Find a way to assign 0/1 (false/true) to the variables such that the formula evaluates to true, that is each clause evaluates to true.
2. Pick a literal from each clause and find a truth assignment to make all of them true. You will fail if two of the literals you pick are in conflict, i.e., you pick x_i and $\neg x_i$

We will take the second view of **3SAT** to construct the reduction.

Interpreting **3SAT**

There are two ways to think about **3SAT**

1. Find a way to assign 0/1 (false/true) to the variables such that the formula evaluates to true, that is each clause evaluates to true.
2. Pick a literal from each clause and find a truth assignment to make all of them true. You will fail if two of the literals you pick are in conflict, i.e., you pick x_i and $\neg x_i$

We will take the second view of **3SAT** to construct the reduction.

Interpreting **3SAT**

There are two ways to think about **3SAT**

1. Find a way to assign 0/1 (false/true) to the variables such that the formula evaluates to true, that is each clause evaluates to true.
2. Pick a literal from each clause and find a truth assignment to make all of them true. You will fail if two of the literals you pick are in conflict, i.e., you pick x_i and $\neg x_i$

We will take the second view of **3SAT** to construct the reduction.

The Reduction

1. G_φ will have one vertex for each literal in a clause
2. Connect the 3 literals in a clause to form a triangle; the independent set will pick at most one vertex from each clause, which will correspond to the literal to be set to true
3. Connect 2 vertices if they label complementary literals; this ensures that the literals corresponding to the independent set do not have a conflict
4. Take k to be the number of clauses

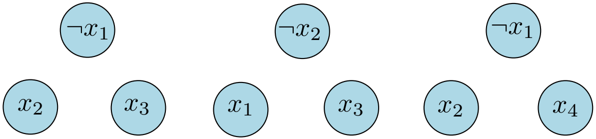

Figure: Graph for $\varphi = (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4)$

The Reduction

1. G_φ will have one vertex for each literal in a clause
2. Connect the 3 literals in a clause to form a triangle; the independent set will pick at most one vertex from each clause, which will correspond to the literal to be set to true
3. Connect 2 vertices if they label complementary literals; this ensures that the literals corresponding to the independent set do not have a conflict
4. Take k to be the number of clauses

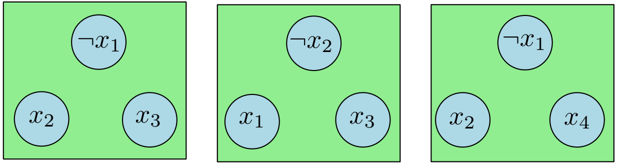

Figure: Graph for $\varphi = (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4)$

The Reduction

1. G_φ will have one vertex for each literal in a clause
2. Connect the 3 literals in a clause to form a triangle; the independent set will pick at most one vertex from each clause, which will correspond to the literal to be set to true
3. Connect 2 vertices if they label complementary literals; this ensures that the literals corresponding to the independent set do not have a conflict
4. Take k to be the number of clauses

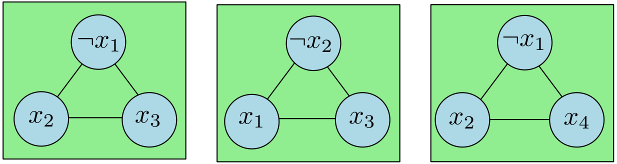

Figure: Graph for $\varphi = (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4)$

The Reduction

1. G_φ will have one vertex for each literal in a clause
2. Connect the 3 literals in a clause to form a triangle; the independent set will pick at most one vertex from each clause, which will correspond to the literal to be set to true
3. Connect 2 vertices if they label complementary literals; this ensures that the literals corresponding to the independent set do not have a conflict
4. Take k to be the number of clauses

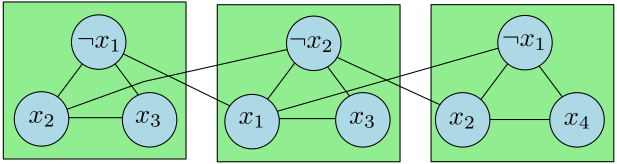

Figure: Graph for $\varphi = (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4)$

The Reduction

1. G_φ will have one vertex for each literal in a clause
2. Connect the 3 literals in a clause to form a triangle; the independent set will pick at most one vertex from each clause, which will correspond to the literal to be set to true
3. Connect 2 vertices if they label complementary literals; this ensures that the literals corresponding to the independent set do not have a conflict
4. Take k to be the number of clauses

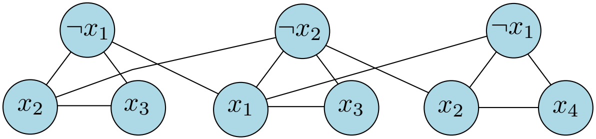

Figure: Graph for $\varphi = (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4)$

Correctness

Proposition 23.2.

φ is satisfiable iff G_φ has an independent set of size k ($=$ number of clauses in φ).

Proof.

$\Rightarrow$ Let a be the truth assignment satisfying φ

- Pick one of the vertices, corresponding to true literals under a, from each triangle. This is an independent set of the appropriate size. Why?

□

Correctness

Proposition 23.2.

φ is satisfiable iff $\boldsymbol{G}_\varphi$ has an independent set of size $\boldsymbol{k}$ (= number of clauses in φ).

Proof.

$\Rightarrow$ Let $\boldsymbol{a}$ be the truth assignment satisfying φ

- Pick one of the vertices, corresponding to true literals under $\boldsymbol{a}$, from each triangle. This is an independent set of the appropriate size. Why?

□

Correctness

Proposition 23.2.

φ is satisfiable iff G_φ has an independent set of size k ($=$ number of clauses in φ).

Proof.

$\Leftarrow$ Let S be an independent set of size k

1. S must contain exactly one vertex from each clause
2. S cannot contain vertices labeled by conflicting literals
3. Thus, it is possible to obtain a truth assignment that makes in the literals in S true; such an assignment satisfies one literal in every clause □

Summary

Theorem 23.3.

Independent set *is* **NP-Complete** *(i.e.,* **NPC***).*

THE END

...

(for now)