1.2
Countable sets, countably infinite sets, and languages

Countable sets

Definition

A set X is countable, if its elements can be counted.
There exists an injective mapping from X to natural numbers $\mathrm{N} = \{1, 2, 3, \ldots\}$.

Example

All finite sets are countable: $\{aba, ima, saba, safta, uma, upa\}$.

Example

$\mathbb{N} \times \mathbb{N} = \{(i, j) \mid i, j \in \mathbb{N}\}$ is countable.

: Proof: $f(i, j) = 2^i 3^j$.

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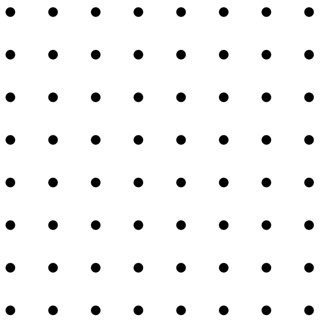

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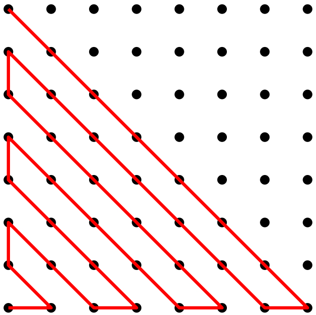

Canonical order and countability of strings

Definition

A set X is countably infinite (countable and infinite) if there is a bijection f between the natural numbers and X.

Alternatively: X is countably infinite if X is an infinite set and there enumeration of elements of X.

The set of all strings is countable

Theorem

Σ^* is countable for any finite Σ.

Enumerate strings in order of increasing length and for each given length enumerate strings in dictionary order (based on some fixed ordering of Σ).

Example: $\{0, 1\}^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, \ldots\}$.
$\{a, b, c\}^* = \{\epsilon, a, b, c, aa, ab, ac, ba, bb, bc, \ldots\}$

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Exercise I

Question: Is $\Sigma^* \times \Sigma^* = \{(x, y) \mid x, y \in \Sigma^*\}$ countable?

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Exercise II

Answer the following questions taking $\Sigma = \{0, 1\}$.

1. Is a finite set countable?
2. X is countable, and the set $Y \subseteq X$, then is the set Y countable?
3. If X and Y are countable, is $X \setminus Y$ countable?
4. Are all infinite sets countably infinite?
5. If X_i is a countable infinite set, for $i = 1, \ldots, 700$, is $\cup_i X_i$ countable infinite?
6. If X_i is a countable infinite set, for $i = 1, \ldots,$, is $\cup_i X_i$ countable infinite?
7. Let X be a countable infinite set, and consider its power set

$$2^X = \{Y \mid Y \subseteq x\}.$$

The statement "the set 2^X is countable" is correct?

THE END

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(for now)