Prove that each of the following problems is NP-hard.

1. Recall that a 5-coloring of a graph G is a function that assigns each vertex of G a "color" from the set $\{0, 1, 2, 3, 4\}$, such that for any edge uv, vertices u and v are assigned different "colors". A 5-coloring is ***careful*** if the colors assigned to adjacent vertices are not only distinct, but differ by more than $1 \pmod 5$. Prove that deciding whether a given graph has a careful 5-coloring is NP-hard.

Solution: We prove that careful 5-coloring is NP-hard by reduction from the standard 5Color problem.

Given a graph G, we construct a new graph H by replacing each edge in G with a path of length three. I claim that H has a careful 5-coloring if and only if G has a (not necessarily careful) 5-coloring.

$\Longleftarrow$ Suppose G has a 5-coloring. Consider a single edge uv in G, and suppose *color*$(u) = a$ and *color*$(v) = b$. We color the path from u to v in H as follows:

- If $b = (a + 1) \bmod 5$, use colors $(a,\ (a + 2) \bmod 5,\ (a - 1) \pmod 5,\ b)$.
- If $b = (a - 1) \bmod 5$, use colors $(a,\ (a - 2) \bmod 5,\ (a + 1) \pmod 5,\ b)$.
- Otherwise, use colors $(a,\ b,\ a,\ b)$.

In particular, every vertex in G retains its color in H. The resulting 5-coloring of H is careful.

$\Longrightarrow$ On the other hand, suppose H has a careful 5-coloring. Consider a path (u, x, y, v) in H corresponding to an arbitrary edge uv in G. There are exactly eight careful colorings of this path with *color*$(u) = 0$, namely: $(0, 2, 0, 2)$, $(0, 2, 0, 3)$, $(0, 2, 4, 1)$, $(0, 2, 4, 2)$, $(0, 3, 0, 3)$, $(0, 3, 0, 2)$, $(0, 3, 1, 3)$, $(0, 3, 1, 4)$. It follows immediately that *color*$(u) \neq$ *color*(v). Thus, if we color each vertex of G with its color in H, we obtain a valid 5-coloring of G.

Given G, we can clearly construct H in polynomial time. ■

2. Prove that the following problem is NP-hard: Given an undirected graph G, find *any* integer $k > 374$ such that G has a proper coloring with k colors but G does not have a proper coloring with $k - 374$ colors.

   Solution: Let G' be the union of 374 copies of G, with additional edges between *every* vertex of each copy and *every* vertex in *every* other copy. Given G, we can easily build G' in polynomial time by brute force. Let $\chi(G)$ and $\chi(G')$ denote the minimum number of colors in any proper coloring of G, and define $\chi(G')$ similarly.

   $\Longrightarrow$ Fix any coloring of G with $\chi(G)$ colors. We can obtain a proper coloring of G' with $374 \cdot \chi(G)$ colors, by using a distinct set of $\chi(G)$ colors in each copy of G. Thus, $\chi(G') \leq 374 \cdot \chi(G)$.

   $\Longleftarrow$ Now fix any coloring of G' with $\chi(G')$ colors. Each copy of G in G' must use its own distinct set of colors, so at least one copy of G uses at most $\lfloor \chi(G')/374 \rfloor$ colors. Thus, $\chi(G) \leq \lfloor \chi(G')/374 \rfloor$.

   These two observations immediately imply that $\chi(G') = 374 \cdot \chi(G)$. It follows that if k is an integer such that $k - 374 < \chi(G') \leq k$, then $\chi(G) = \chi(G')/374 = \lceil k/374 \rceil$. Thus, if we could compute such an integer k in polyomial time, we could compute $\chi(G)$ in polynomial time. But computing $\chi(G)$ is NP-hard! ■

3. A ***bicoloring*** of an undirected graph assigns each vertex a set of *two* colors. There are two types of bicoloring: In a *weak* bicoloring, the endpoints of each edge must use *different* sets of colors; however, these two sets may share one color. In a *strong* bicoloring, the endpoints of each edge must use *distinct* sets of colors; that is, they must use four colors altogether. Every strong bicoloring is also a weak bicoloring.

   (a) Prove that finding the minimum number of colors in a weak bicoloring of a given graph is NP-hard.

   Solution: It suffices to prove that deciding whether a graph has a weak bicoloring with three colors is NP-hard, using the following trivial reduction from the standard 3COLOR problem.

   Let G be an arbitrary undirected graph. I claim that G has a proper 3-coloring if and only if G has a weak bicoloring with 3 colors.

   - Suppose G has a proper coloring using the colors red, green, and blue. We can obtain a weak bicoloring of G using only the colors cyan, magenta, and yellow by recoloring each red vertex with $\{\text{magenta}, \text{yellow}\}$, recoloring each blue vertex with $\{\text{magenta}, \text{cyan}\}$, and recoloring each green vertex with $\{\text{yellow}, \text{cyan}\}$.
   - Suppose G has a weak bicoloring using the colors cyan, magenta, yellow. Then we can obtain a proper 3-coloring of G by defining red $= \{\text{magenta}, \text{yellow}\}$, defining blue $= \{\text{magenta}, \text{cyan}\}$, and defining green $= \{\text{yellow}, \text{cyan}\}$.

   More generally, for any integer k and any graph G, every weak k-bicoloring of G is also a proper $\binom{k}{2}$-coloring of G, and vice versa. ■

(b) Prove that finding the minimum number of colors in a strong bicoloring of a given graph is NP-hard.

Solution: It suffices to prove that deciding whether a graph has a strong bicoloring with six colors is NP-hard, using the following reduction from the standard 3Color problem.

Let G be an arbitrary undirected graph. We build a new graph H from G as follows:

- For every vertex v in G, the graph H contains three vertices v_1, v_2, and v_3 and three edges v_1v_2, v_2v_3, and v_3v_1.
- For every edge uv in G, the graph H contains three edges u_1v_1, u_2v_2, and u_3v_3.

I claim that G has a proper 3-coloring if and only if H has a strong bicoloring with six colors. Without loss of generality, we can assume that G (and therefore H) is connected; otherwise, consider each component independently.

$\Rightarrow$ Suppose G has a proper 3-coloring with colors red, green, and blue. Then we define a strong bicoloring of H with colors $1, 2, 3, 4, 5, 6$ as follows:
  - For every red vertex v in G, let *color*$(v_1) = \{1, 2\}$ and *color*$(v_1) = \{3, 4\}$ and *color*$(v_3) = \{5, 6\}$.
  - For every blue vertex v in G, let *color*$(v_1) = \{3, 4\}$ and *color*$(v_1) = \{5, 6\}$ and *color*$(v_3) = \{1, 2\}$.
  - For every green vertex v in G, let *color*$(v_1) = \{5, 6\}$ and *color*$(v_1) = \{1, 2\}$ and *color*$(v_3) = \{3, 4\}$.

  Exhaustive case analysis confirms that every pair of adjacent vertices of H has disjoint color sets.

- Suppose H has a strong bicoloring with six colors. Fix an arbitrary vertex v in G, and without loss of generality, suppose *color*$(v_1) = \{1, 2\}$ and *color*$(v_1) = \{3, 4\}$ and *color*$(v_3) = \{5, 6\}$. Exhaustive case analysis implies that for any edge uv, each vertex u_i must be colored either $\{1, 2\}$ or $\{3, 4\}$ or $\{5, 6\}$. It follows by induction that *every* vertex in H must be colored either $\{1, 2\}$ or $\{3, 4\}$ or $\{5, 6\}$.

  Now for each vertex w in G, color w red if *color*$(w_1) = \{1, 2\}$, blue if *color*$(w_1) = \{3, 4\}$, and green if *color*$(w_1) = \{5, 6\}$. This assignment of colors is a proper 3-coloring of G.

Given G, we can build H in polynomial time by brute force. ■

I believe that deciding whether a graph has a strong bicoloring with five *colors is also NP-hard, but I don't have a proof yet. A graph has a strong bicoloring with four colors if and only if it is bipartite, and a strong bicoloring with two or three colors if and only if it has no edges.*