1. Recall the following kColor problem: Given an undirected graph G, can its vertices be colored with k colors, so that every edge touches vertices with two different colors?

    (a) Describe a direct polynomial-time reduction from 3Color to 4Color.

    Solution: Suppose we are given an arbitrary graph G. Let H be the graph obtained from G by adding a new vertex a (called an *apex*) with edges to every vertex of G. I claim that G is 3-colorable if and only if H is 4-colorable.

    $\Longrightarrow$ Suppose G is 3-colorable. Fix an arbitrary 3-coloring of G, and call the colors "red", "green", and "blue". Assign the new apex a the color "plaid". Let uv be an arbitrary edge in H.

    - If both u and v are vertices in G, they have different colors.
    - Otherwise, one endpoint of uv is plaid and the other is not, so u and v have different colors.

    We conclude that we have a valid 4-coloring of H, so H is 4-colorable.

    $\Longleftarrow$ Suppose H is 4-colorable. Fix an arbitrary 4-coloring; call the apex's color "plaid" and the other three colors "red", "green", and "blue". Each edge uv in G is also an edge of H and therefore has endpoints of two different colors. Each vertex v in G is adjacent to the apex and therefore cannot be plaid. We conclude that by deleting the apex, we obtain a valid 3-coloring of G, so G is 3-colorable.

    We can easily transform G into H in polynomial time by brute force. ■

(b) Prove that kColor problem is NP-hard for any $k \geq 3$.

Solution (direct): The lecture notes include a proof that 3Color is NP-hard. For any integer $k > 3$, I'll describe a direct polynomial-time reduction from 3Color to kColor.

Let G be an arbitrary graph. Let H be the graph obtain from G by adding $k-3$ new vertices $a_1, a_2, \ldots, a_{k-3}$, each with edges to every other vertex in H (including the other a_i's). I claim that G is 3-colorable if and only if H is k-colorable.

$\Longrightarrow$ Suppose G is 3-colorable. Fix an arbitrary 3-coloring of G. Color the new vertices $a_1, a_2, \ldots, a_{k-3}$ with $k-3$ new distinct colors. Every edge in H is either an edge in G or uses at least one new vertex a_i; in either case, the endpoints of the edge have different colors. We conclude that H is k-colorable.

$\Longleftarrow$ Suppose H is k-colorable. Each vertex a_i is adjacent to every other vertex in H, and therefore is the only vertex of its color. Thus, the vertices of G use only three distinct colors. Every edge of G is also an edge of H, so its endpoints have different colors. We conclude that the induced coloring of G is a proper 3-coloring, so G is 3-colorable.

Given G, we can construct H in polynomial time by brute force. ■

Solution (induction): Let k be an arbitrary integer with $k \geq 3$. Assume that jColor is NP-hard for any integer $3 \leq j < k$. There are two cases to consider.

- If $k = 3$, then kColor is NP-hard by the reduction from 3Sat in the lecture notes.
- Suppose $k = 3$. The reduction in part (a) directly generalizes to a polynomial-time reduction from $(k-1)$Color to kColor: To decide whether an arbitrary graph G is $(k-1)$-colorable, add an apex and ask whether the resulting graph is k-colorable. The induction hypothesis implies that $(k-1)$Color is NP-hard, so the reduction implies that kColor is NP-hard.

In both cases, we conclude that kColor is NP-hard. ■

2. A *Hamiltonian cycle* in a graph G is a cycle that goes through every vertex of G exactly once. Deciding whether an arbitrary graph contains a Hamiltonian cycle is NP-hard.

   A ***tonian cycle*** in a graph G is a cycle that goes through at least *half* of the vertices of G. Prove that deciding whether a graph contains a tonian cycle is NP-hard.

Solution (duplicate the graph): I'll describe a polynomial-time reduction from HamiltonianCycle. Let G be an arbitrary graph. Let H be a graph consisting of two disjoint copies of G, with no edges between them; call these copies G_1 and G_2. I claim that G has a Hamiltonian cycle if and only if H has a tonian cycle.

$\Longrightarrow$ Suppose G has a Hamiltonian cycle C. Let C_1 be the corresponding cycle in G_1. C_1 contains exactly half of the vertices of H, and thus is a tonian cycle in H.

$\Longleftarrow$ On the other hand, suppose H has a tonian cycle C. Because there are no edges between the subgraphs G_1 and G_2, this cycle must lie entirely within one of these two subgraphs. G_1 and G_2 each contain exactly half the vertices of H, so C must also contain exactly half the vertices of H, and thus is a *Hamiltonian* cycle in either G_1 or G_2. But G_1 and G_2 are just copies of G. We conclude that G has a Hamiltonian cycle.

Given G, we can construct H in polynomial time by brute force. ■

Solution (add n new vertices): I'll describe a polynomial-time reduction from HamiltonianCycle. Let G be an arbitrary graph, and suppose G has n vertices. Let H be a graph obtained by adding n new vertices to G, but no additional edges. I claim that G has a Hamiltonian cycle if and only if H has a tonian cycle.

$\Longrightarrow$ Suppose G has a Hamiltonian cycle C. Then C visits exactly half the vertices of H, and thus is a tonian cycle in H.

$\Longleftarrow$ On the other hand, suppose H has a tonian cycle C. This cycle cannot visit any of the new vertices, so it must lie entirely within the subgraph G. Since G contains exactly half the vertices of H, the cycle C must visit every vertex of G, and thus is a Hamiltonian cycle in G.

Given G, we can construct H in polynomial time by brute force. ■

To think about later:

3. Let G be an undirected graph with weighted edges. A Hamiltonian cycle in G is ***heavy*** if the total weight of edges in the cycle is at least half of the total weight of all edges in G. Prove that deciding whether a graph contains a heavy Hamiltonian cycle is NP-hard.

Solution (two new vertices): I'll describe a polynomial-time a reduction from the Hamiltonian ***path*** problem. Let G be an arbitrary undirected graph (without edge weights). Let H be the edge-weighted graph obtained from G as follows:

- Add two new vertices s and t.
- Add edges from s and t to all the other vertices (including each other).
- Assign weight 1 to the edge st and weight 0 to every other edge.

The total weight of all edges in H is 1. Thus, a Hamiltonian cycle in H is heavy if and only if it contains the edge st. I claim that H contains a heavy Hamiltonian cycle if and only if G contains a Hamiltonian path.

$\Longrightarrow$ First, suppose G has a Hamiltonian path from vertex u to vertex v. By adding the edges vs, st, and tu to this path, we obtain a Hamiltonian cycle in H. Moreover, this Hamiltonian cycle is heavy, because it contains the edge st.

$\Longleftarrow$ On the other hand, suppose H has a heavy Hamiltonian cycle. This cycle must contain the edge st, and therefore must visit all the other vertices in H contiguously. Thus, deleting vertices s and t and their incident edges from the cycle leaves a Hamiltonian path in G.

Given G, we can easily construct H in polynomial time by brute force. ■

Solution (smartass): I'll describe a polynomial-time a reduction from the standard Hamiltonian cycle problem. Let G be an arbitrary graph (without edge weights). Let H be the edge-weighted graph obtained from G by assigning each edge weight 0. I claim that H contains a heavy Hamiltonian cycle if and only if G contains a Hamiltonian path.

$\Longrightarrow$ Suppose G has a Hamiltonian cycle C. The total weight of C is at least half the total weight of all edges in H, because $0 \geq 0/2$. So C is a heavy Hamiltonian cycle in H.

$\Longleftarrow$ Suppose H has a heavy Hamiltonian cycle C. By definition, C is also a Hamiltonian cycle in G.

Given G, we can easily construct H in polynomial time by brute force. ■