1. Let $L = \{w \in \{a, b\}^* \mid$ an a appears in some position i of w, and a b appears in position $i+2\}$.

   (a) Create an NFA N for L with at most four states.

   (b) Using the "power-set" construction, create a DFA M from N. Rather than writing down the sixteen states and trying to fill in the transitions, build the states as needed, because you won't end up with unreachable or otherwise superfluous states.

   (c) **Work on this later:** Now directly design a DFA M' for L with only five states, and explain the relationship between M and M'.

   For the rest of the problems assume that L is an arbitrary regular language.

2. Prove that the language $\mathit{reverse}(L) := \{w^R \mid w \in L\}$ is regular. *Hint:* Consider a DFA M that accepts L and construct a NFA that accepts $\mathit{reverse}(L)$.

3. Prove that the language $\mathit{insert}\mathtt{1}(L) := \{x\mathtt{1}y \mid xy \in L\}$ is regular.

   Intuitively, $\mathit{insert}\mathtt{1}(L)$ is the set of all strings that can be obtained from strings in L by inserting exactly one $\mathtt{1}$. For example, if $L = \{\varepsilon, \mathtt{OOK!}\}$, then $\mathit{insert}\mathtt{1}(L) = \{\mathtt{1}, \mathtt{1OOK!}, \mathtt{O1OK!}, \mathtt{OO1K!}, \mathtt{OOK1!}, \mathtt{OOK!1}\}$.

Work on these later:

4. Prove that the language $\mathit{delete}\mathtt{1}(L) := \{xy \mid x\mathtt{1}y \in L\}$ is regular.

   Intuitively, $\mathit{delete}\mathtt{1}(L)$ is the set of all strings that can be obtained from strings in L by deleting exactly one $\mathtt{1}$. For example, if $L = \{\mathtt{101101}, \mathtt{00}, \varepsilon\}$, then $\mathit{delete}\mathtt{1}(L) = \{\mathtt{01101}, \mathtt{10101}, \mathtt{10110}\}$.

5. Consider the following recursively defined function on strings:

$$\mathit{stutter}(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ aa \bullet \mathit{stutter}(x) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

   Intuitively, $\mathit{stutter}(w)$ doubles every symbol in w. For example:

   - $\mathit{stutter}(\mathtt{PRESTO}) = \mathtt{PPRREESSTTOO}$
   - $\mathit{stutter}(\mathtt{HOCUS{\diamond}POCUS}) = \mathtt{HHOOCCUUSS{\diamond}{\diamond}PPOOCCUUSS}$

   Let L be an arbitrary regular language.

   (a) Prove that the language $\mathit{stutter}^{-1}(L) := \{w \mid \mathit{stutter}(w) \in L\}$ is regular.

   (b) Prove that the language $\mathit{stutter}(L) := \{\mathit{stutter}(w) \mid w \in L\}$ is regular.

6. Consider the following recursively defined function on strings:

$$evens(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ \varepsilon & \text{if } w = a \text{ for some symbol } a \\ b \cdot evens(x) & \text{if } w = abx \text{ for some symbols } a \text{ and } b \text{ and some string } x \end{cases}$$

Intuitively, $evens(w)$ skips over every other symbol in w. For example:

- $evens(\texttt{EXPELLIARMUS}) = \texttt{XELAMS}$
- $evens(\texttt{AVADA}\diamond\texttt{KEDAVRA}) = \texttt{VD}\diamond\texttt{EAR}$.

Once again, let L be an arbitrary regular language.

(a) Prove that the language $evens^{-1}(L) := \{w \mid evens(w) \in L\}$ is regular.

(b) Prove that the language $evens(L) := \{evens(w) \mid w \in L\}$ is regular.