Greedy Algorithms

Lecture16

Backtracking

- We have seen Backtracking/DP so far

— Make a simple choice

— Recursively solve everything else

e.g. **Subset Sum** : is a certain element of the set in the subset or not? If only we could know…

Backtracking

- We have seen Backtracking/DP so far

Try all options for

— ~~Make~~ a simple choice

— Recursively solve everything else For each choice!

e.g. **Subset Sum** : is a certain element of the set in the subset or not? If only we could know…

LIS: Do I include an element in the sequence or not?

NFA accept: should I transition to a certain state?

(see nondeterminism)

Backtracking

- We have seen Backtracking/DP so far

Try all options for

— ~~Make~~ a simple choice

— Recursively solve everything else For each choice!

Greedy

Really tempting to

- Choose one option
- Recurse (e.g. Edit Distance: choose two characters that are equal to leave them as such)

Course Policy on Greedy

- When you use greedy algorithm, you need to ALWAYS prove correctness. Otherwise you get a zero, EVEN IF THE ALGORITHM IS CORRECT!
- Greedy is a loaded gun!

Greedy Algorithm Example

- Sorting files on magnetic tape (not RAM)
- Remember music cassettes?
- Blue Water Supercomputer.

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The Problem:

- Given an array of lengths of each file: L[1…n]
- I want to sort the files so that if someone asks me for a random file, the expected time it takes to wind the tape to the start of the file and rewind it back is small.

Sorting Files on Tape

The Problem:

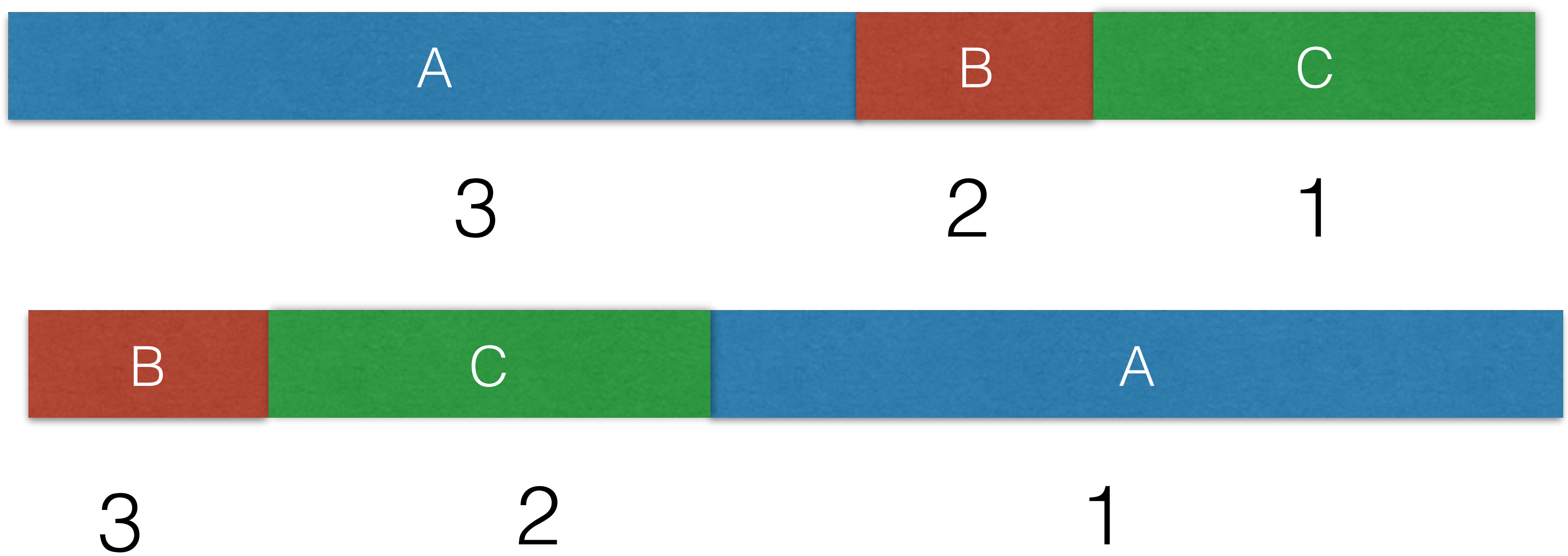
- Given an array of lengths of each file: L[1…n]
- I want to sort the files so that if someone asks me for a random file, the expected time it takes to wind the tape to the start of the file and rewind it back is small.
- Formally, I want to find a permutation that minimizes

$$\sum_{k=1}^{n}\sum_{i=1}^{k} L[\pi(i)]$$

Where $\pi(i)$ is the index of the file sorted in position i of the tape

Sorting Files on Tape

What order should I sort them?



Claim:

Sort L, in increasing order of lengths is the best solution

$$L[\pi(i)] \leq L[\pi(i+1)] \quad \text{for all } i$$

Needs proof!!!

Sorting Files on Tape

Proof:

Assume in optimal ordering π

$$L[\pi(i)] > L[\pi(i+1)] \quad \text{for some} \quad i$$

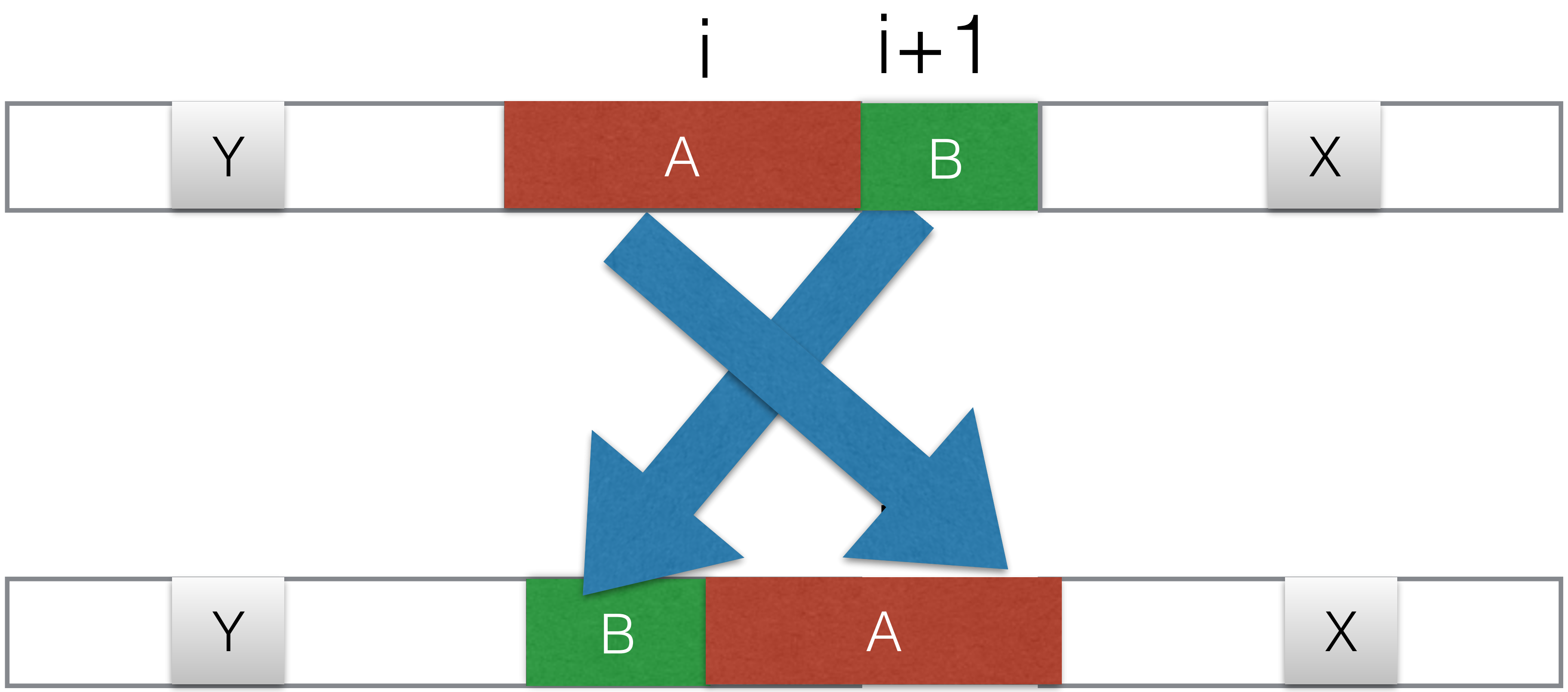
what happens if we switch A and B?

Sorting Files on Tape

Proof:

Assume in optimal ordering π

$$L[\pi(i)] > L[\pi(i+1)] \quad \text{for some} \quad i$$



Cost(A) increases by L[B]
Cost(B) decreases by L[A]

Total cost increases by L[B]-L[A] <0

Exchange Argument

- Consider any non-greedy solution
- Perform an exchange to make the solution look more greedy
- Argue that the new solution after doing the exchange is no worse.
- In out example, the new solution was strictly better, so greedy is the only way.

Sorting Files on Tape

- What if I also had frequencies?
- L[1…n] lengths of files and F[1…n] frequencies.
- Need to minimize: $\displaystyle\sum_{k=1}^{n}\sum_{i=1}^{k}(F[\pi(k)]\cdot L[\pi(i)])$
- If all the lengths the same and frequencies different?

Punchline: Swapping adjacent files A,B increases cost by L[B]F[A]-L[A]F[B] <0

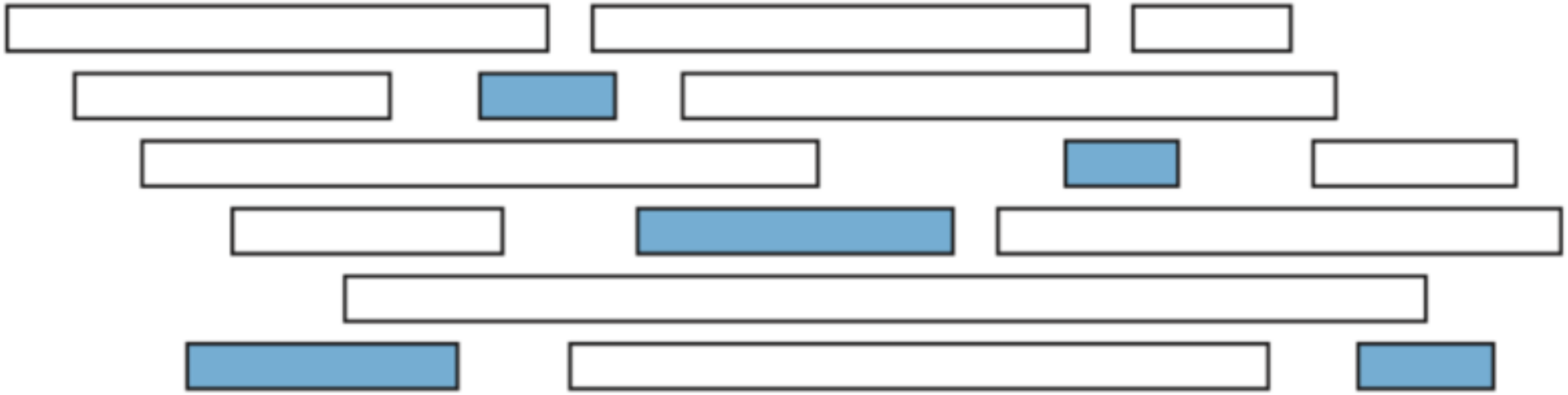
L F

Class Scheduling

- University decides to start a new major, CS+ climbing
- Degree requirements involve taking certain number of classes, certain hours and certain categories.
- Bulk of the degree is determined by taking a certain number of classes. None of these classes require actual work.
- Without the instructors permission, you cannot register for two classes whose times overlap.
- You only need to sign up! Goal: sign up for as many classes as possible, without overlapping classes.

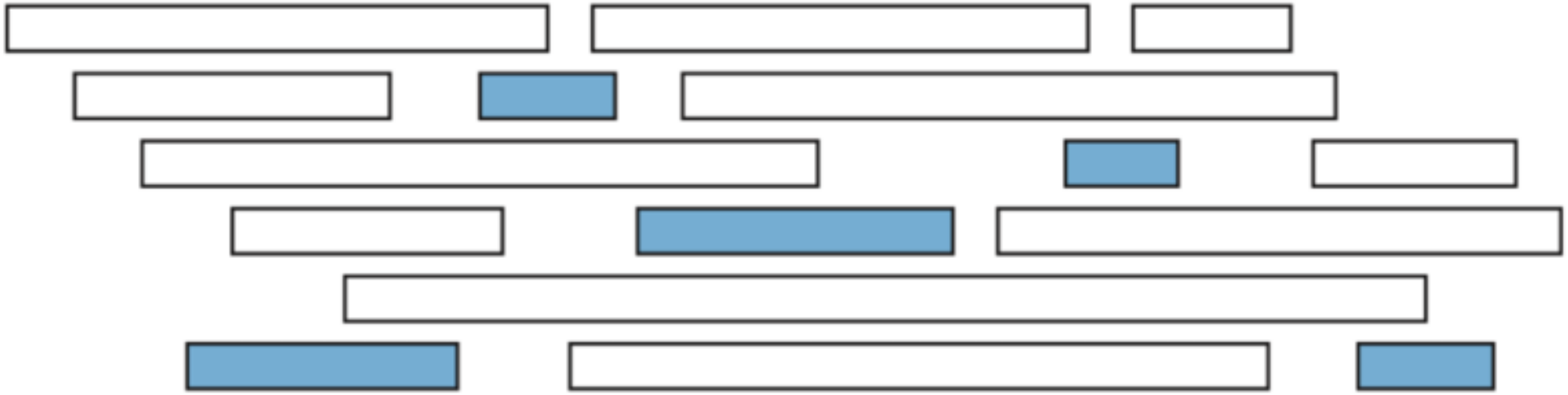
Class Scheduling

- Given a collection of intervals with start and end time, want to choose a subset of those intervals such that no pair overlaps.
- Subset needs to be as large as possible.
- Model it as a graph problem (next time): Independent Set!



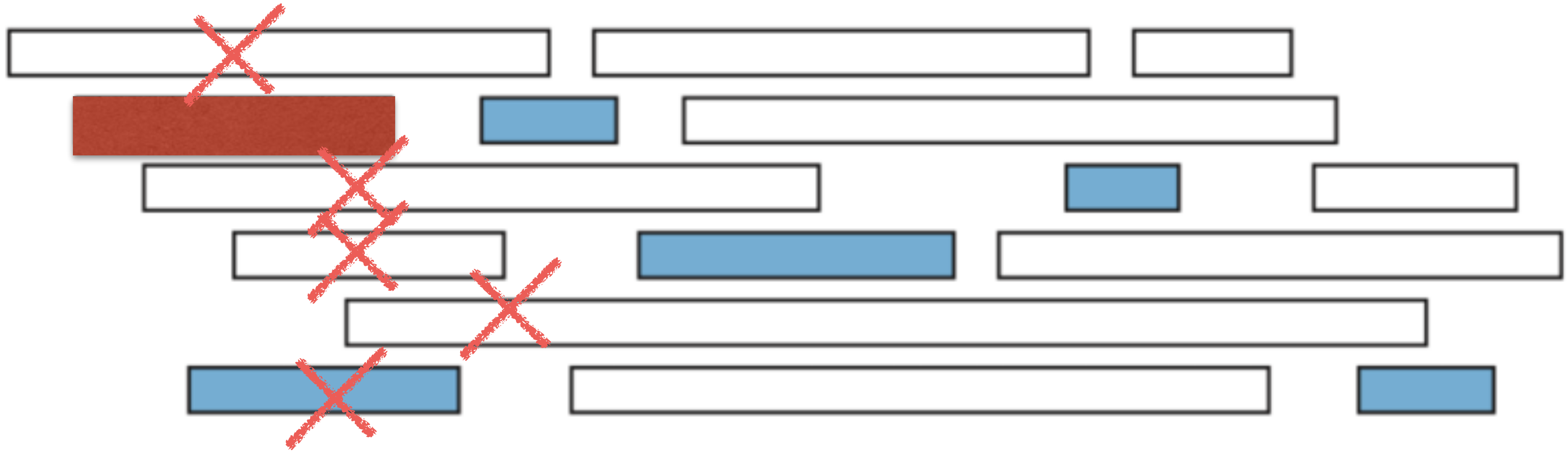
Class Scheduling

- Algorithm? DP? Greedy?
- e.g. find the earliest class, take it and recurse
- find the longest class, throw it away and recurse.
- find the shortest class, take it and recurse.



Class Scheduling

- None of those work!
- Instead: pick the class the ends earliest



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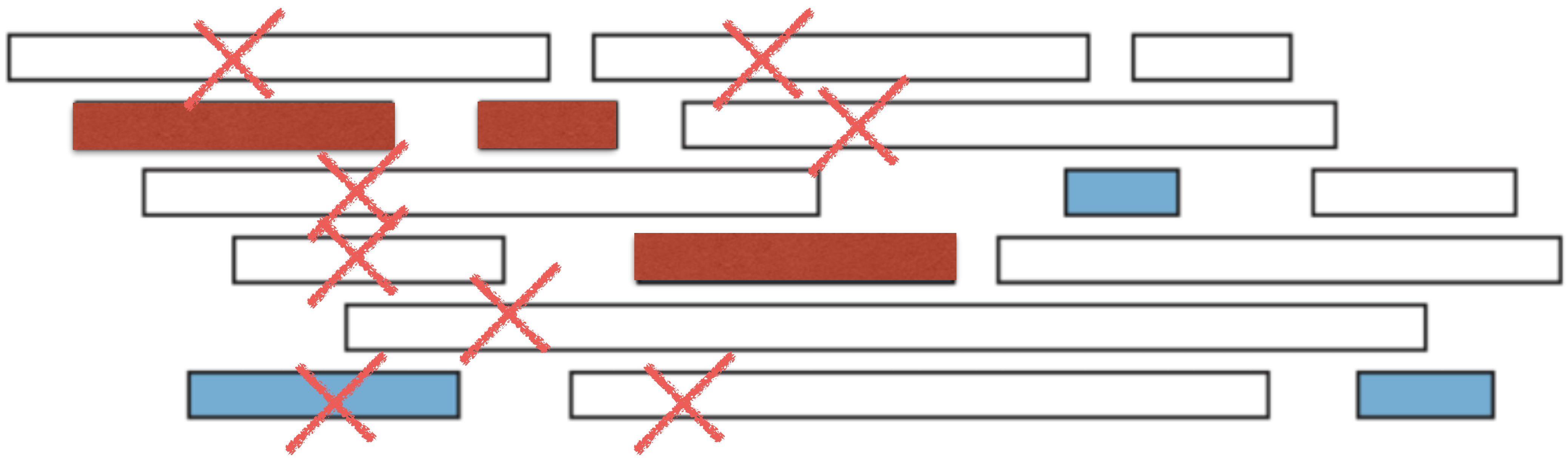

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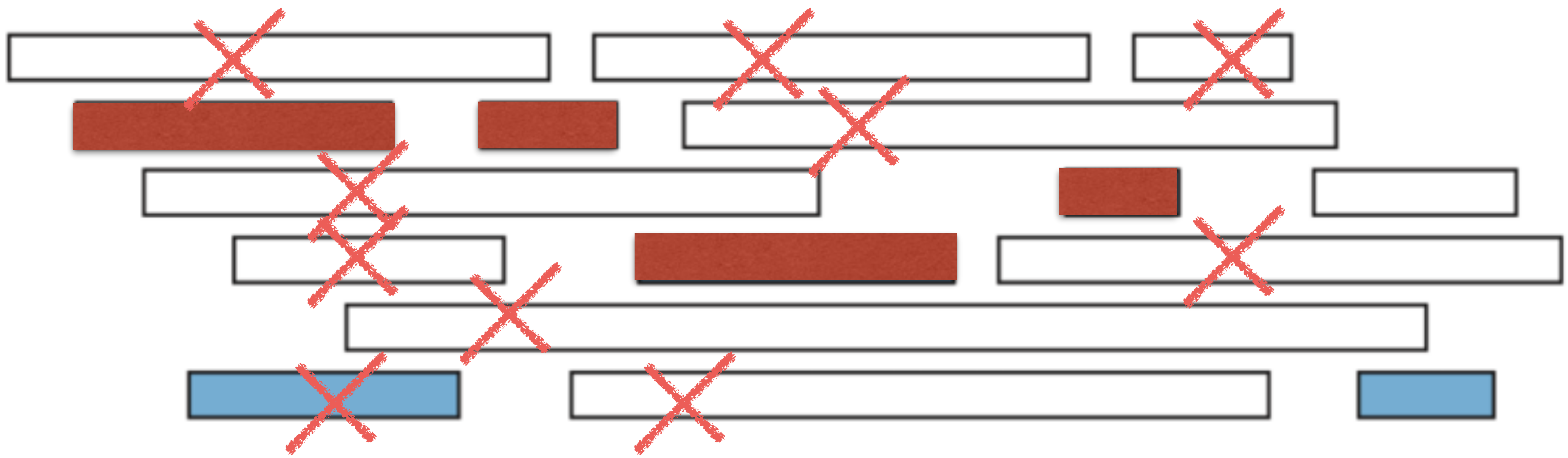

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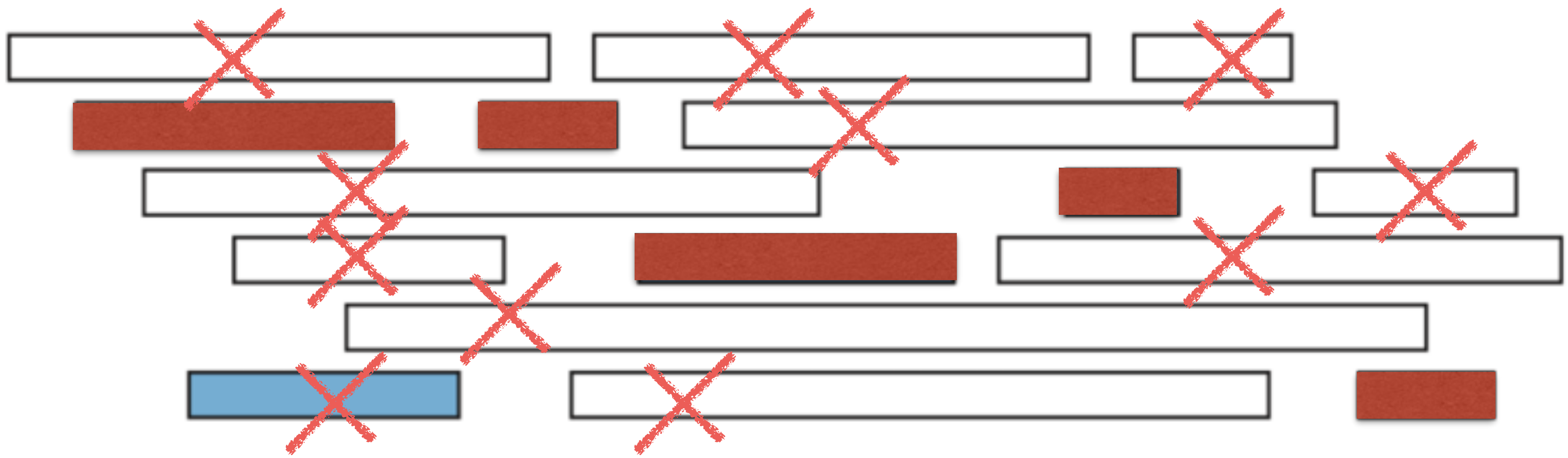
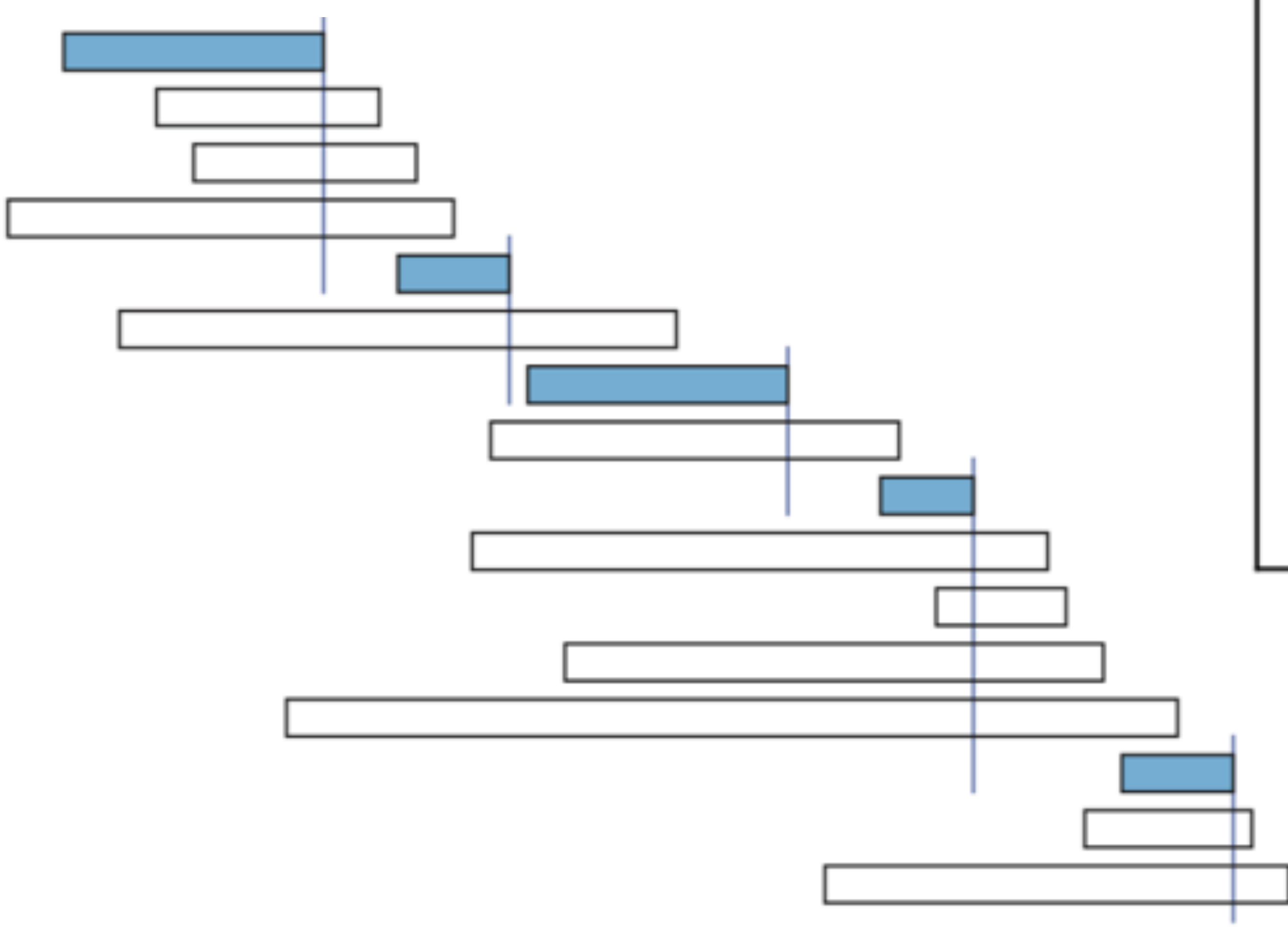


Class Scheduling

- Sort classes according to finish time

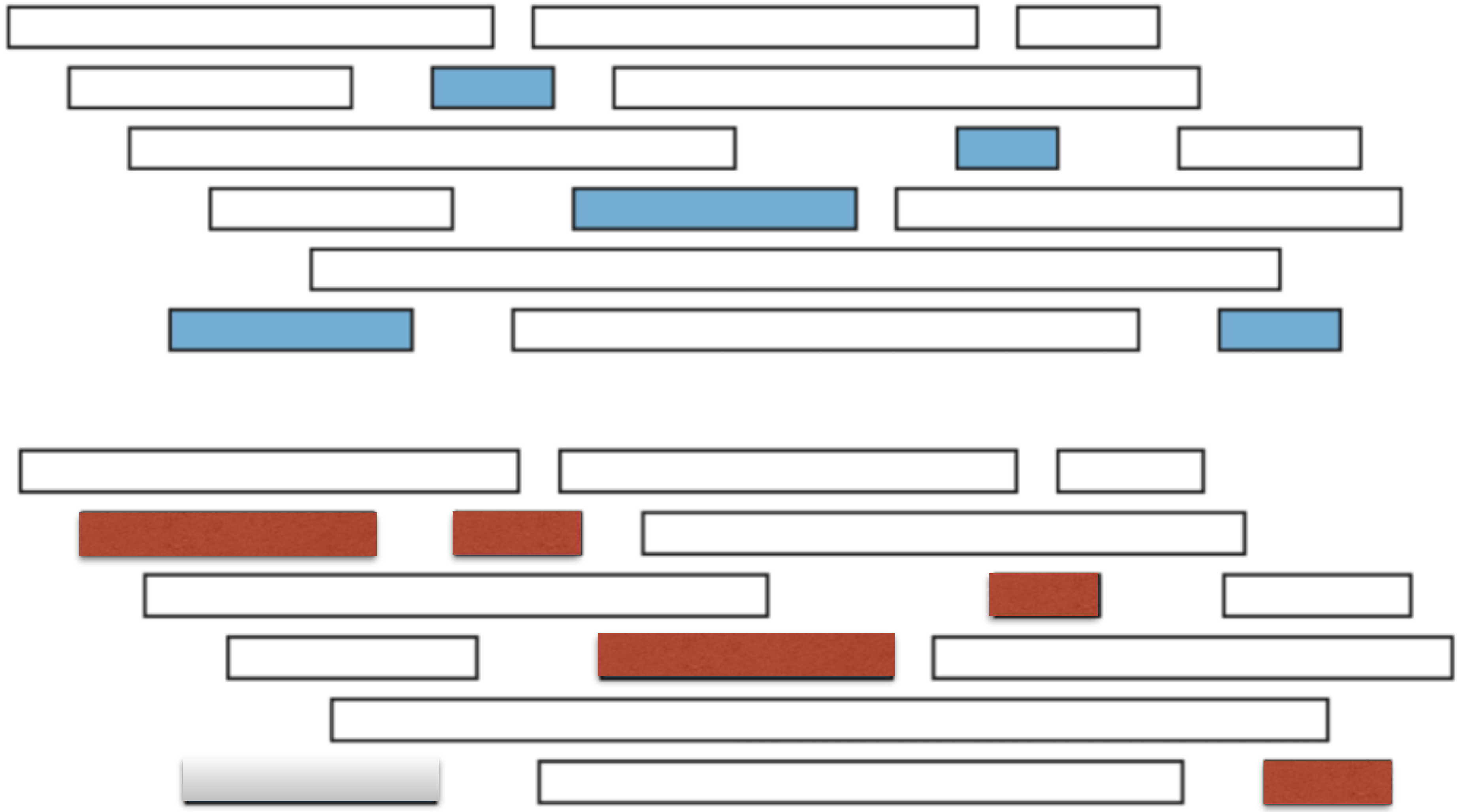


```
GREEDYSCHEDULE(S[1..n], F[1..n]):
    sort F and permute S to match
    count ← 1
    X[count] ← 1
    for i ← 2 to n
        if S[i] > F[X[count]]
            count ← count + 1
            X[count] ← i
    return X[1..count]
```

Because of sorting, O(nlogn), while DP in $O(n^2)$

Class Scheduling

- Why is it optimal? Proof!
- Not the only optimal schedule. There are many optimal schedules.



Class Scheduling

- Exchange argument.
- Think of it as a recursive algorithm. Pick the class what finishes first and then recurse.
- Proof by induction!

Lemma:

At least one maximal conflict free schedule includes the class that ends first.

Class Scheduling

Lemma:

At least one maximal conflict free schedule includes the class that ends first.

Proof:

Let f be the class that ends first.

Consider any schedule X that excludes f.

Let g be the first class ending in X.

F[f]<F[g] implies

that f does not overlap any class in X\{g}

$\boxed{Y = X - \{g\} + \{f\}}$ is a valid schedule of same size!

What if X is empty?

Huffman Codes

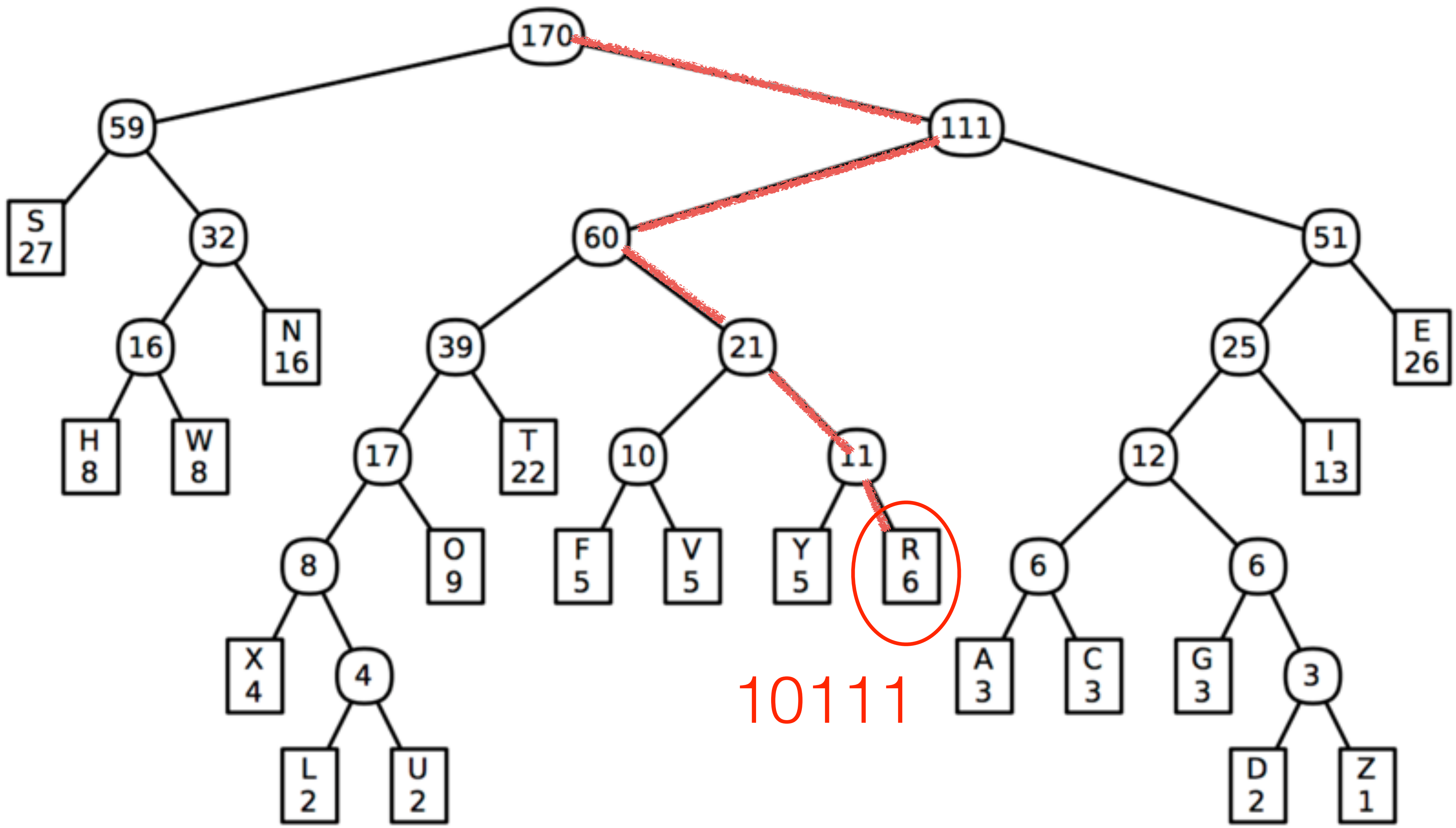
- Binary code assigns a string of 0s and 1s to each character in the alphabet.
- 7-bit ASCII code, Unicode, Morse
- We want the code to be prefix free (Morse code is not).
- Any prefix free code can be visualized as a binary code tree, where the characters are stored at the leafs.
- Codeword for each symbol is given by the path from the root to the corresponding leaf (e.g 1 for right 0 for left).
- Length of codeword for a symbol is the depth of the corresponding leaf.

Huffman Codes

- Goal is to encode messages in an n-character alphabet so that the encoded message is as short as possible.
- Given array of frequencies: f[1…n], we want to compute a prefix-free binary code that minimizes the total encoded length of message.

$$\sum_{i=1}^{n} f[i] \cdot \mathrm{depth}(i)$$

Huffman Codes



Huffman Codes

This sentence contains three a's, three c's, two d's, twenty-six e's, five f's, three g's, eight h's, thirteen i's, two l's, sixteen n's, nine o's, six r's, twenty-seven s's, twenty-two t's, two u's, five v's, eight w's, four x's, five y's, and only one z.

A	C	D	E	F	G	H	I	L	N	O	R	S	T	U	V	W	X	Y	Z
3	3	2	26	5	3	8	13	2	16	9	6	27	22	2	5	8	4	5	1

Huffman's algorithm: merge two least frequent letters and recurse!

Huffman Codes

A	C	D	E	F	G	H	I	L	N	O	R	S	T	U	V	W	X	Y	Z
3	3	2	26	5	3	8	13	2	16	9	6	27	22	2	5	8	4	5	1

A	C	E	F	G	H	I	L	N	O	R	S	T	U	V	W	X	Y	DZ
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Huffman Codes

A	C	E	F	G	H	I	L	N	O	R	S	T	U	V	W	X	Y	DZ
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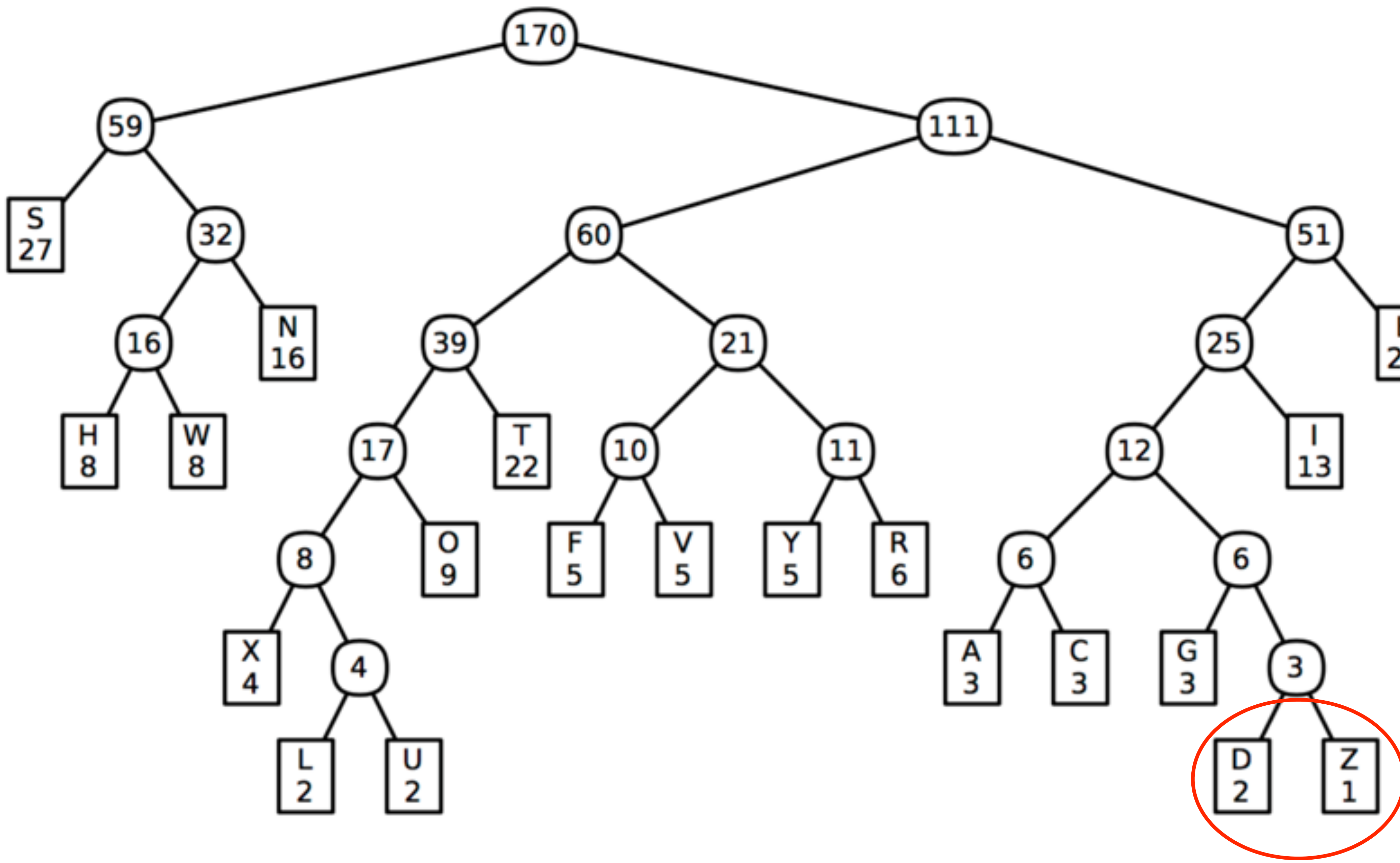

Huffman Codes

Lemma: Let x and y be the two least frequent characters. There is an optimal code tree in which x and y are siblings, and have the largest depth of any leaf.

Proof: Exchange argument!

Assume, for the optimal schedule that the deepest two leaves are not x and y.

```
BUILDHUFFMAN(f[1..n]):
  for i ← 1 to n
      L[i] ← 0;  R[i] ← 0
      INSERT(i, f[i])
  for i ← n to 2n − 1
      x ← EXTRACTMIN()
      y ← EXTRACTMIN()
      f[i] ← f[x] + f[y]
      L[i] ← x;  R[i] ← y
      P[x] ← i;  P[y] ← i
      INSERT(i, f[i])
  P[2n − 1] ← 0
```

```
HUFFMANENCODE(A[1..k]):
  m ← 1
  for i ← 1 to k
      HUFFMANENCODEONE(A[i])

HUFFMANENCODEONE(x):
  if x < 2n − 1
      HUFFMANENCODEONE(P[x])
      if x = L[P[x]]
          B[m] ← 0
      else
          B[m] ← 1
      m ← m + 1
```

```
HUFFMANDECODE(B[1..m]):
  k ← 1
  v ← 2n − 1
  for i ← 1 to m
      if B[i] = 0
          v ← L[v]
      else
          v ← R[v]
      if L[v] = 0
          A[k] ← v
          k ← k + 1
          v ← 2n − 1
```